

TESE DE DOUTORAMENTO

**NONLINEAR DIFFERENTIAL
EQUATIONS ON BOUNDED
AND UNBOUNDED DOMAINS**

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ESCOLA DE DOUTORAMENTO INTERNACIONAL
PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

ANO 2018



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Nonlinear differential equations on bounded and unbounded domains

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Nonlinear differential equations on bounded and unbounded domains

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To Marcos





Acknowledgements

As a final highlight of the work conducting to this Thesis, I would like to thank all those people who, in one way or another, have made it possible.

First of all, as it could not be any in other way, I would like to thank my supervisor, Alberto Cabada, for the inestimable support that he has provided me, both professionally and personally, during the more than five years that we have been working together.

I would also like to show my gratitude to Professors Rachůnková and Feliz Minhós for their warm welcome during my stays in Olomouc and Évora. Working with them has been a really enriching experience for me.

I acknowledge all my colleagues of Mathematical Analysis at University of Santiago de Compostela, for their good personal treatment during these years.

I would like to especially mention Adrián, for being always willing to lend a hand and helping me to solve many of my crises (and also for all the coffee liquor shots).

I can not forget my fellows, Ignacio, Jorge and Lorena, for having made the work something enjoyable. It is likely that without our gossiping sessions this Thesis would have been finished before, but I'm sure it would not have been as great as it was.

Thanks to my family, especially to my parents, Deme and Aurora. You have always supported me, both in my professional and personal decisions, and I know for sure that I could have never reached this without you. Thank you for being always there.

I also want to thank my brother Pablo, for being the person who better knows me. I really hope that you never stop trying to make me smile with your bad jokes.

Thanks to Marcos, for accompanying me throughout this adventure and all the rest. Thank you simply for being this way, "intrepid".

I would also like to thank my lifetime friends, for staying by my side even if they think that I am crazy: to Iago, for understanding me sometimes better than anyone else and for being the best travel guide; to Lucía, for being that constant in my life and for taking all the necessary photos to put it on record; to Yaiza, for begin a breath of fresh air and for stopping thinking that I am cold and calculating; to María, for always having the right words to say and for having chased a famous' double once with me; to Pita and Álvaro, because even if they are far away, they made every reunion worthwhile.

I could not finish without mentioning my flatmates, who have been able to tolerate me during these last months and have made me feel at home: Suso, who always has the perfect song for every moment, and Javi, with whom I have shared many great colacaos.

Finally, thank you to all the rest of my doctorate mates from the Faculty, especially to Franco, Andrea and Arís, for having been part of this beautiful stage that now finishes.

Como punto final a esta Tesis, me gustaría dar las gracias a todos los que, de un modo u otro, han hecho que esto sea posible.

En primer lugar, como no podía ser menos, quiero agradecer a mi director de Tesis, Alberto Cabada, el inestimable apoyo, tanto en lo profesional como en lo personal, que ha sido para mí en estos más de cinco años que llevamos trabajando juntos.

Agradezco asimismo a los profesores Irena Rachůnková y Feliz Minhós por su cálida acogida durante mis estancias en Olomouc y Évora. Ha sido una experiencia muy enriquecedora poder trabajar con ellos.

Me gustaría agradecer también a todos mis compañeros de Análisis Matemático de la Universidad de Santiago de Compostela, por la buena acogida y el buen trato recibido en estos años.

Quiero mencionar especialmente a Adrián, por estar siempre dispuesto a echar una mano y haberme ayudado a resolver muchas de mis crisis (y también por todos esos chupitos de licor café).

No puedo olvidar tampoco a mis compañeros Ignacio, Jorge y Lorena, por haber conseguido que el trabajo fuese también algo divertido. Es muy probable que sin nuestras sesiones de cotilleo esta Tesis se hubiese acabado antes, pero estoy segura de que hubiese merecido mucho menos la pena.

Gracias a mi familia, especialmente a mis padres, Deme y Aurora. Me habéis apoyado siempre, tanto en mis decisiones profesionales como en las personales, y sé que no hubiese llegado hasta aquí sin vosotros. Gracias por estar siempre ahí.

Quiero darle las gracias también a mi hermano Pablo, por ser la persona que mejor me conoce. Espero que nunca dejes de intentar sacarme una sonrisa con tus chistes malos.

Gracias a Marcos, por acompañarme en esta aventura y en todas las demás. Gracias simplemente por ser así, “intrépido”.

Me gustaría agradecer también a mis amigos, los de siempre, por seguir estando a mi lado pese a pensar que estoy loca: a Iago, por entenderme a veces mejor que nadie y por ser la mejor guía de viajes; a Lucía, por ser esa constante en mi vida y por hacer todas las fotos necesarias para que quede constancia; a Yaiza, por ser un soplo de aire fresco y porque ya no piensa que soy fría y calculadora; a María, por saber decir siempre las palabras adecuadas y porque una vez corrimos detrás del doble de un famoso; a Pita y Álvaro, porque aunque están lejos, hacen que cada reencuentro valga la pena.

No podrían faltar aquí mis compañeros de piso, que han sido capaces de aguantarme estos últimos meses y me han hecho sentirme en casa: Suso, que siempre tiene la canción perfecta para cada momento, y Javi, con quien he compartido muchos colacaos memorables.

Gracias también a todos los demás compañeros de doctorado de la facultad, especialmente a Franco, Andrea y Arís, por haber formado parte de esta etapa tan bonita que llega a su fin.

*Lucía López Somoza
22 de noviembre de 2018*

“The case for my life, then, or for that of any one else who has been a mathematician in the same sense which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.”

G. H. Hardy, *A Mathematician's Apology*





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Glossary

\mathbb{N} : Set of natural numbers, that is, $\{1, 2, \dots\}$.

\mathbb{Z} : Set of integer numbers.

\mathbb{R} : Set of real numbers.

\mathbb{R}^+ : Set of positive real numbers.

$\overline{\mathbb{R}}$: Extended real line, that is $\overline{\mathbb{R}} = [-\infty, \infty]$.

$\mathcal{C}(I) \equiv \mathcal{C}(I, \mathbb{R})$: Space of continuous real functions defined on an interval I .

$\mathcal{C}^n(I) \equiv \mathcal{C}^n(I, \mathbb{R})$, $n \in \mathbb{N}$: Space of n -times differentiable real functions defined on an interval I such that the j -th derivative is continuous for $j = 0, \dots, n$.

$\mathcal{C}^\infty(I) \equiv \mathcal{C}^\infty(I, \mathbb{R})$: Space of infinitely differentiable real functions defined on an interval I .

$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$: Space of n -times differentiable real functions with real limits at $\pm\infty$, that is,

$$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) = \left\{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} : f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}), \exists \lim_{t \rightarrow \pm\infty} f^{(j)}(t) \in \mathbb{R}, j = 0, \dots, n \right\}.$$

$\tilde{\mathcal{C}}_\varphi^n$: Space of continuously n -differentiable φ -extensions to infinity:

$$\tilde{\mathcal{C}}_\varphi^n \equiv \tilde{\mathcal{C}}_\varphi^n(\mathbb{R}, \mathbb{R}) = \left\{ f \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}) : \exists \tilde{f} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}), f = \varphi \tilde{f}|_{\mathbb{R}} \right\}.$$

$L^\alpha(I)$, $1 \leq \alpha < \infty$: Space of the measurable functions f on the interval I such that the Lebesgue integral of $|f|^\alpha$ is finite.

$\|f\|_\alpha$, $1 \leq \alpha < \infty$: Norm of f in the space $L^\alpha(I)$, that is,

$$\|f\|_\alpha = \left(\int_I |f(t)|^\alpha dt \right)^{\frac{1}{\alpha}}.$$

$L^\infty(I)$: Space of the measurable functions f on the interval I such that are essentially bounded.

$\|f\|_\infty$: Norm of f in the space $L^\infty(I)$, that is,

$$\|f\|_\infty = \sup \{|f(t)|, t \in I\}.$$

$\mathcal{AC}(I)$: Space of absolutely continuous functions, that is,

$$\mathcal{AC}(I) = \left\{ u \in \mathcal{C}(I) : \exists f \in L^1(I), u(t) = u(t_0) + \int_{t_0}^t f(s) \, ds, t, t_0 \in I \right\}.$$

$W^{k,p}(I)$, $k, p \in \mathbb{N}$: Sobolev space $k - p$ on the set I , that is,

$$W^{k,p}(I) = \left\{ u \in \mathcal{C}^{k-1}(I) : u^{(k-1)} \in \mathcal{AC}(I), u^{(k)} \in L^p(I) \right\}.$$

a_+ : Positive part of $a : X \rightarrow \mathbb{R}$, that is, $a_+(t) = \max\{a(t), 0\}$.

a_- : Negative part of $a : X \rightarrow \mathbb{R}$, that is, $a_-(t) = \max\{-a(t), 0\}$.

α^* : Conjugate of the real number $\alpha \geq 1$, that is, the real number such that $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$. If $\alpha = 1$ then $\alpha^* = \infty$ and vice-versa.

$a \succ 0$: $a \in L^\alpha(I)$ such that $a(t) \geq 0$ for a. e. $t \in I$ and $a \not\equiv 0$ on I .

$\overline{\Omega}$: Closure of the set Ω .

$\partial\Omega$: Boundary of the set Ω .

$i_K(\mathcal{T}, \Omega)$: Index of operator \mathcal{T} relative to Ω in the cone K .

$r(\mathcal{T})$: Spectral radius of operator \mathcal{T} .

$\mu(\mathcal{T})$: Principal characteristic value of operator \mathcal{T} .

Preface

Differential equations represent one of the strongest connections between Mathematics and real life. This is due to the fact that almost all the physical phenomena, as well as many other in economy, biology or chemistry, are modelled by differential equations. This seems thus a good reason to dedicate our efforts to trying to solve this kind of problems.

In particular, we will analyze the qualitative properties of the solutions of nonlinear equations, focusing on the study of constant sign solutions on the whole domain of definition or, at least, on some subset of it. The interest of this property is due to the fact that many of the physical magnitudes which appear in differential problems can not take negative values (typical examples would be pressure, power or temperature in Kelvin degrees). Moreover, in many problems in engineering, models study the deviation of certain structures from their equilibrium point. In this context, if we want to maintain the structure stable, the deformation must occur always in the same direction, which, mathematically speaking, means that the solution must have constant sign.

The most common techniques to ensure the existence of solution for these problems are based on the construction of an abstract formulation included into functional analysis, in which the solutions of the differential equations coincide with either the fixed points or the critical ones of certain operators. In this Thesis, we will work with the first method, constructing integral operators which will be determined by some kernel related to the linear part of the equation. This kernel is the so-called Green's function.

In many of the cases, operators will be defined in subsets called cones, which will let us transfer the properties of the Green's function to the solutions of the considered problem. This fact lets us intuit now the importance that the study of the properties of linear problems (and, specially of Green's functions) has on the research concerning nonlinear ones.

The present Thesis is divided into two parts, which deal with differential problems on bounded and unbounded domains, respectively. It contains most of the work developed by the author in the last years. All these discoveries appear in several

publications which the reader may consult, namely [22, 23, 27, 31–34, 102, 103, 131].



Summary

The present Thesis, compiled under the title “*Nonlinear differential equations on bounded and unbounded domains*”, contains almost the whole work developed by the author during the last years.

It is divided into two parts: the first of them, which comprises six chapters, covers the study of boundary value problems defined on bounded intervals, as well as the more general case of Hammerstein integral equations. The second part, which comprises three chapters, is focused on the study of both differential and integral problems defined on unbounded domains.

It should be noted that, although the title only mentions nonlinear differential equations, the first chapters of the Thesis will be devoted to the study of linear boundary value problems. This is due to the fact that the properties of these linear problems, and particularly those of the related Green’s function, will determine the best way to approach the search for solutions of nonlinear problems.

We include now a brief summary of the main results given in each chapter.

Chapter 1: Preliminary Results

For the purpose of writing a self-contained work, this chapter compiles some preliminary results which will be used throughout the remaining of this Thesis.

First, in Section 1.1, we introduce the definition and main properties of the Green’s function. As we will see, this function is a very powerful tool to study both linear and nonlinear differential problems. This is due to the fact that every differential problem can be transformed into an equivalent integral one of which the kernel is, precisely, the aforementioned Green’s function.

This way, the problem of finding solutions of differential problems will naturally lead to the more general framework of finding fixed points of integral operators. It is in this context where the results ensuring the existence of fixed points of arbitrary compact operators defined in Banach spaces acquire great importance. Some of these results are collected in Section 1.2, namely the very well-known Schauder’s Fixed Point Theorem and the classical fixed point index theory (which, following the line

of [64], is introduced for arbitrary open sets, which might be unbounded).

Finally, another important tool that we will use in this Thesis to study properties of linear operators is spectral theory. In particular, this theory combined with the fixed point index results, makes it possible to prove the existence of solutions of certain integral problems. Basic results regarding spectral theory are compiled in Section 1.3.

Chapter 2: Green's Functions and Spectral Theory for Even Order Linear BVPs

This chapter contains a fully detailed study of even order linear boundary value problems. In particular, we study problems related to the following operator coupled with various boundary conditions:

$$L u(t) \equiv u^{(2n)}(t) + a_{2n-1}(t) u^{(2n-1)}(t) + \cdots + a_1(t) u'(t) + a_0(t) u(t), \quad t \in I \equiv [0, T],$$

where $a_k : I \rightarrow \mathbb{R}$, $a_k \in L^\alpha(I)$, $\alpha \geq 1$, $k = 0, \dots, 2n-1$.

From this operator we will define two more, concretely

$$\tilde{L} u(t) \equiv u^{(2n)}(t) + \hat{a}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{a}_{2n-2}(t) u^{(2n-2)}(t) + \cdots + \hat{a}_1(t) u'(t) + \tilde{a}_0(t) u(t), \quad t \in J \equiv [0, 2T],$$

where \tilde{a}_{2k} is the even extension of a_{2k} to the interval J and \hat{a}_{2k+1} is the odd extension of a_{2k+1} to J , for $k = 0, \dots, n-1$, and

$$\tilde{\tilde{L}} u(t) \equiv u^{(2n)}(t) + \hat{\hat{a}}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{\tilde{a}}_{2n-2}(t) u^{(2n-2)}(t) + \cdots + \hat{\hat{a}}_1(t) u'(t) + \tilde{\tilde{a}}_0(t) u(t), \quad t \in [0, 4T],$$

where $\tilde{\tilde{a}}_{2k}$ and $\hat{\hat{a}}_{2k+1}$ are the even and odd extensions to the interval $[0, 4T]$ of \tilde{a}_{2k} and \hat{a}_{2k+1} , respectively, for $k = 0, \dots, n-1$.

The main idea of this chapter consists of expressing the Green's function of each Neumann, Dirichlet and mixed problems related to operator L as a sum of Green's functions of periodic and antiperiodic problems related to \tilde{L} . This way, the following equalities are proved:

$$\begin{aligned} G_N[T](t, s) &= G_P[2T](t, s) + G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_D[T](t, s) &= G_P[2T](t, s) - G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_{M_1}[T](t, s) &= G_A[2T](t, s) - G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_{M_2}[T](t, s) &= G_A[2T](t, s) + G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \end{aligned}$$

where $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ denote, respectively, the Green's functions of Neumann, Dirichlet and mixed problems related to operator L . Analogously, $G_P[2T]$ and $G_A[2T]$ denote the Green's functions of periodic and antiperiodic problems related to \tilde{L} .

Something similar can be done to decompose all the previous Green's functions as a linear combination of the one related to the periodic problem associated to \tilde{L} evaluated in different points.

Since the Green's function is a fundamental tool for studying both linear and nonlinear problems, being able to relate different Green's functions will let us relate also the spectra and the solutions of the different problems.

First, the previous expressions provide a direct relation between the spectra of the considered problems. In particular, we deduce various decompositions of some spectra as the union of others. Moreover, we also obtain a certain order relation between the first eigenvalues of each problem.

On the other hand, we are also able to deduce that the constant sign of one Green's function implies the same constant sign of another one. This can be seen in the following result.

Corollary 1 (Corollary 2.4.1). *The following properties hold for any coefficients $a_0, \dots, a_{2n-1} \in L^1(I)$:*

1. *If $G_P[2T] \leq 0$ on $J \times J$, then $G_N[T] \leq 0$ on $I \times I$.*
2. *If $G_P[2T] \geq 0$ on $J \times J$, then $G_N[T] \geq 0$ on $I \times I$.*
3. *If $G_N[2T] \leq 0$ on $J \times J$, then $G_N[T] \leq 0$ on $I \times I$.*
4. *If $G_N[2T] \geq 0$ on $J \times J$, then $G_N[T] \geq 0$ on $I \times I$.*
5. *If $G_D[2T] \leq 0$ on $J \times J$, then $G_{M_2}[T] \leq 0$ on $I \times I$.*
6. *If $G_D[2T] \geq 0$ on $J \times J$, then $G_{M_2}[T] \geq 0$ on $I \times I$.*

With respect to previous corollary, it must be pointed out that it can be improved for order $n = 1$, something that will be done in Chapter 3. On the other hand, it is proved in this chapter that the converse of Assertions 1 and 2 holds when all of the coefficients a_0, \dots, a_{2n-1} are constants, whereas the converse of the other assertions does not hold, not even in the constant case, for $n > 1$. Moreover, a counterexample is given to show that the converse of Assertion 2 is not true in general for $n > 1$. Finally, it remains as an open problem to see if Assertion 1 is an equivalence or not when $n > 1$.

Finally, in Section 2.5, under the assumption of constant sign of some Green's function, we obtain some point by point inequalities between two different Green's

functions. This lets us deduce that the solution of the problem under certain boundary conditions is smaller at every point than the solution of another problem with the same operator but different boundary conditions.

The results in this chapter are compiled in [31].

Chapter 3: Second Order Equation

This chapter considers the problem studied in Chapter 2 in the particular case of the second order equation (that is, we will take $n = 1$).

The reason why this case is studied independently from the general one is the fact that, when working with second order differential equations, it is possible to use Sturm-Liouville's theory. This theory, which does not hold for differential equations of higher order, provides some properties of oscillation of the solutions of the equations. This will let us obtain stronger results than in previous chapter.

In this chapter, two different problems are considered. First, in Section 3.2, we study the problem related to Hill's operator

$$L u(t) \equiv u''(t) + a(t) u(t), \quad t \in I.$$

This will be a particular case of operator L given in Chapter 3 for $n = 1$ and $a_1 \equiv 0$. We note that the fact of considering $a_1 \equiv 0$ is not an important loss of generality in the results as every second order differential equation written in the form

$$u''(t) + a_1(t) u'(t) + a_0(t) u(t) = 0,$$

can be transformed into a Hill's equation through a suitable change of variable, as long as the coefficients a_0 and a_1 have enough regularity.

The results obtained in this section are more powerful than the corresponding ones given in Chapter 3. An example which illustrates this is the following theorem in which we relate the constant sign of different Green's functions.

Theorem 2 (Theorem 3.2.22). *For any $a \in L^1(I)$ the following properties hold:*

1. $G_P[2T] < 0$ on $J \times J$ if and only if $G_N[T] < 0$ on $I \times I$. This is equivalent to $G_N[2T] < 0$ on $J \times J$.
2. $G_P[2T] > 0$ on $(0, 2T) \times (0, 2T)$ if and only if $G_N[T] > 0$ on $(0, T) \times (0, T)$.
3. If $G_N[2T] > 0$ on $(0, 2T) \times (0, 2T)$ then $G_N[T] > 0$ on $(0, T) \times (0, T)$.
4. If $G_P[2T] < 0$ on $J \times J$ then $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$.
5. If $G_P[2T] > 0$ on $(0, 2T) \times (0, 2T)$ then $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$.

6. If $G_N[T]$ (or, equivalently, $G_P[2T]$) has constant sign on $I \times I$, then $G_D[T] < 0$ on $(0, T) \times (0, T)$, $G_{M_1}[T] < 0$ on $[0, T) \times [0, T)$ and $G_{M_2}[T] < 0$ on $(0, T] \times (0, T]$.
7. $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$ if and only if $G_{M_2}[T] < 0$ on $(0, T) \times (0, T]$.
8. If either $G_{M_2}[T] < 0$ on $(0, T] \times (0, T]$ or $G_{M_1}[T] < 0$ on $[0, T) \times [0, T)$, then $G_D[T] < 0$ on $(0, T) \times (0, T)$.

In the same way, the point by point inequalities between various Green's functions are also more precise, which implies that we obtain more precision when it comes to compare the solutions of different problems. Thus, while in the previous chapter we were only able to ensure that the solution of a problem was smaller at every point than the solution of another one, now we will ensure also that both solutions have constant sign.

Moreover, whereas in the previous chapter we could only establish a certain order relation between the first eigenvalues of each problem, we prove in this chapter an alternation between all the eigenvalues of all the considered problems.

Finally, we consider some explicit criteria to ensure the constant sign of the Green's function of the periodic problem and, using the relations between different Green's function, we will adapt them to the rest of the considered boundary value problems.

On the other hand, Section 3.3 deals with the general second order equation given in self-adjoint form, namely

$$(p u')'(t) + \bar{a}(t) u(t) = \bar{\sigma}(t), \quad \text{a.e. } t \in I,$$

with $p > 0$ a.e. $t \in I$, $\frac{1}{p} \in L^1(I)$ and \bar{a} and $\bar{\sigma}$ such that $\bar{a} p^{\frac{\alpha-1}{\alpha}}, \bar{\sigma} p^{\frac{\alpha-1}{\alpha}} \in L^\alpha(I)$, for some $\alpha \in [1, \infty]$.

We prove in this section that the Green's function of any boundary value problem related to the previous equation can be expressed in terms of the Green's function related to the Hill's operator coupled with the same boundary conditions. As a consequence, all the results obtained in the previous section can be adapted to this more general framework.

This chapter collects results from [22] and [23].

Chapter 4: Solutions for Even Order Nonlinear BVPs with Constant Sign Green's Functions

This chapters considers, for the first time in this Thesis, nonlinear boundary value problems.

In particular, we will consider nonlinear problems that fulfil the following scheme

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X,$$

being L the $2n$ -th order general linear operator defined in Chapter 3.

On the other hand, we will consider $X \subset W^{2n,1}(I)$, a Banach space where the boundary conditions are included and on which L is nonresonant.

Under these conditions, it occurs that the solutions of the previous boundary value problem correspond to the fixed points in X of the following integral operator

$$L^{-1} u(t) = \int_0^T G[T](t, s) f(s, u(s)) \, ds,$$

begin $G[T]$ the related Green's function.

The technique that we use to ensure the existence of fixed points of this integral operator is the lower and upper solutions method.

The main novelty of our approach with respect to previous works is the fact that we are able to ensure the existence of solution of the problem using a pair of lower and upper solutions of a different problem (composed by the same operator coupled with different boundary conditions). This is possible thanks to the point by point relations between Green's functions which were proved in Chapters 2 and 3.

Moreover, it must be pointed out that one of the basic hypotheses for this chapter is the constant sign of the Green's functions.

Results in this chapter can be seen in [31].

Chapter 5: Positive Solutions for Nonlinear Second Order BVPs with Sign-Changing Green's Functions

This chapter is devoted to the study of the existence of constant sign solutions of a boundary value problem of order two related to Hill's operator in the case where, contrary to what happened in the previous chapter, the Green's function changes its sign.

The basic idea behind the method developed in this chapter is based on the fact that, even if the Green's function changes sign, it is possible to ensure that the integral of this function multiplied by the eigenfunction related to the first eigenvalue of the problem is positive.

We will present the following reasoning in terms of the periodic problem, but we note that it is equally valid for any other boundary condition.

Consider then the following periodic boundary value problem

$$\begin{cases} u''(t) + a(t) u(t) = f(t, u(t)), & t \in I, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

and let G_P and v_P be the related Green's function and the eigenfunction associated to the first eigenvalue, respectively. Then it holds that

$$\int_0^T G_P(t, s) v_P(s) \, ds > 0 \quad \text{for all } t \in I,$$

which justifies that the following constant is well-defined

$$\gamma = \inf_{t \in I} \frac{\int_0^T G_P^+(t, s) v_P(s) \, ds}{\int_0^T G_P^-(t, s) v_P(s) \, ds} (> 1).$$

Moreover, consider the following hypotheses:

(H₁) $f: I \times [0, \infty) \rightarrow [0, \infty)$ satisfies L¹-Carathéodory conditions.

(H₂) There exist two positive constants m and M such that

$$m v_P(t) \leq f(t, x) \leq M v_P(t)$$

for every $t \in I$ and $x \geq 0$. Moreover, these constants satisfy that $\frac{M}{m} \leq \gamma$.

(H₃) There exists $[c, d] \subset I$ such that $\int_c^d G_P(t, s) \, dt \geq 0$, for all $s \in I$ and $\int_c^d G_P(t, s) \, dt > 0$, for all $s \in [c, d]$.

Then, if the previous conditions hold and the Green's function changes its sign, it is proved that there exists a solution in the cone

$$K = \left\{ u \in \mathcal{C}(I, \mathbb{R}) : u \geq 0 \text{ on } I, \int_0^T u(s) \, ds \geq \sigma \|u\| \right\},$$

where

$$\sigma = \frac{\eta}{\max_{t, s \in I} \{G_P(t, s)\}},$$

and

$$\eta = \min_{s \in [c, d]} \left\{ \int_c^d G_P(t, s) \, dt \right\} > 0.$$

Note that this solution is nonnegative.

All the results in this chapter are collected in [27].

Chapter 6: Existence and Multiplicity Results for some Generalized Hammerstein Equations with a Parameter

This chapter considers integral problems, defined on Banach spaces, which are called generalized Hammerstein equations.

In particular, we will study the existence and multiplicity of fixed points of the following integral operator

$$\mathcal{T}u(t) = \lambda \int_0^T k(t, s) f(s, u(s), u'(s), \dots, u^{(m)}(s)) \, ds, \quad t \in I, \quad (1)$$

where $\lambda > 0$ is a positive parameter, $k : I \times I \rightarrow \mathbb{R}$ is a kernel function which satisfies certain regularity conditions, m is a positive integer and $f : I \times \mathbb{R}^{m+1} \rightarrow [0, +\infty)$ is an L^1 -Carathéodory function.

This chapter generalizes several results existing in the literature by weaken the conditions that the kernel function must satisfy. Concretely, the kernel and some of its derivatives (not necessarily all of them) will be required to be positive only on some subintervals of I . These subintervals may in fact be degenerated, that is, reduced to a single point.

On the other hand, we will look for kernels for which some derivatives (again, not necessarily all of them) satisfy the following inequalities:

$$\left| \frac{\partial^j k}{\partial t^j}(t, s) \right| \leq \phi_j(s) \quad \text{for all } t \in [c_j, d_j] \text{ and a.e. } s \in I,$$

and

$$\frac{\partial^j k}{\partial t^j}(t, s) \geq \xi_j \phi_j(s) \quad \text{for all } t \in [a_j, b_j] \text{ and a.e. } s \in I,$$

begin ϕ_j some integrable functions and ξ_j some constants. It is worth mentioning that the intervals $[a_j, b_j]$ and $[c_j, d_j]$ must have nonempty intersection but may be different or, even, non comparable.

Under several hypotheses (see (H_1) – (H_7) in Section 6.2), we are able to prove the existence of fixed points of the considered integral operator in the cone

$$K = \left\{ u \in C^m(I, \mathbb{R}) : \begin{array}{l} u^{(i)}(t) \geq 0, \quad t \in [m_i, n_i], \quad i \in J_0; \\ \min_{t \in [a_j, b_j]} u^{(j)}(t) \geq \xi_j \|u^{(j)}\|_{[c_j, d_j]}, \quad j \in J_1 \end{array} \right\},$$

where

$$\|u^{(j)}\|_{[c_j, d_j]} := \max_{t \in [c_j, d_j]} |u^{(j)}(t)|,$$

$J \equiv \{0, 1, \dots, m\}$ and $J_1 \subset J_0 \subset J$, $J_1 \neq \emptyset$. To the best of our knowledge this type of cones is new in the literature.

Regarding the techniques used to prove the existence of fixed points, we use two different ones. First, in Section 6.3, the existence of a fixed point is proved by means of the fixed point index defined on arbitrary open sets (some of which are unbounded). On the other hand, in Section 6.4, we give some results of existence and multiplicity of solutions. These results are also based on the fixed point index, considered now on open and bounded sets.

The main difference between both sections is that the hypotheses the nonlinearity f must satisfy are not the same and, in fact, Section 6.5 shows some examples which prove that both methods are not comparable.

Next, Section 6.6 presents an application of the previous results to guarantee the existence of solution of Dirichlet problems of arbitrary even order

$$\begin{cases} u^{(2n)}(t) = f\left(t, u(t), \dots, u^{(2n-1)}(t)\right), & t \in [0, 1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, \dots, n-1. \end{cases}$$

This study generalizes the previous ones in the literature since in this type of problems a usual hypothesis is that the function f may depend only on the even order derivatives, while in this chapter we admit the dependence on any derivative up to order $2n - 1$.

Finally, Section 6.7 considers the particular case of the following differential problem of third order

$$\begin{cases} -u^{(3)}(t) = \lambda f(t, u(t), u'(t), u''(t)), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \end{cases}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$ are given constants.

Results in this chapter can be found in [32] and [102].

Chapter 7: On Multi-Point Resonant Problems on the Half Line

In this chapter we will consider for the first time a problem defined on an unbounded domain.

Concretely, we will prove the existence of bounded solutions for the following

multi-point boundary value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in [0, \infty), \\ u(0) = 0, \quad u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases}$$

where $\alpha_i > 0$ and $0 = \xi_1 < \dots < \xi_{m-1} < +\infty$. In particular, we will assume that the coefficients α_i satisfy the following condition

$$\sum_{i=1}^{m-1} \alpha_i = 1,$$

which implies that the problem is resonant.

To solve this problem we will consider a modified one (which we will construct by adding new terms at both sides of the equation) which will be equivalent to the first one and nonresonant. This modified problem will be transformed into an integral one of which the fixed points will correspond to the solutions of the initial problem. Concretely, the integral problem with which we will work is

$$Tu(t) = \int_0^\infty G(t, s) (f(s, u(s), u'(s)) + k u'(s) + M u(s)) \, ds,$$

where G denotes the Green's function of the problem

$$\begin{cases} u''(t) + k u'(t) + M u(t) = 0, & t \in [0, \infty), \\ u(0) = 0, \quad u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases}$$

and k and M are two positive constants satisfying certain conditions.

Moreover, the modified problem will satisfy another important problem: its related Green's function will belong to the space $L^1[0, \infty) \cap L^\infty[0, \infty)$. This will imply that the integral operator is compact if the nonlinearity f satisfies either L^1 or L^∞ -Carathéodory conditions. Note that this fact makes it possible to ensure the existence of solution for a bigger set of problems since, as we are working with an unbounded domain, the spaces $L^1[0, \infty)$ and $L^\infty[0, \infty)$ are not comparable.

Finally, to prove the existence of fixed points of the integral operator, we will use the lower and upper solutions method. In particular, to show that the integral operator is compact, we use the compactness criterion given in Theorem 1, which involves a certain equiconvergence condition at infinity.

The results in this chapter are compiled in [103].

Chapter 8: Existence of Solutions of Integral Equations with Asymptotic Conditions

In this chapter we study the fixed points of an integral operator defined on the real line.

In general, the main difficulty when trying to prove the existence of fixed points of integral operators defined on unbounded intervals is being able to prove that the considered operator is compact. These problems are due to the impossibility of applying Ascoli-Arzelà's Theorem to prove the compactness of the operator.

The most common way to solve this problem consists in using a certain compactness criteria (which we have actually used in Chapter 7), which is given in Theorem 1 in page 181.

In this chapter we present an alternative method which will have a double benefit: on the one hand, it will allow us to apply Ascoli-Arzelà's Theorem to prove the compactness of the operator. On the other hand, it will warrant that the solutions of the problem have a particular asymptotic behavior.

To do that, we will define a suitable Banach space including these asymptotic properties. In particular, for $n \in \mathbb{N}$, consider the space of the real functions of class n which have limit at $\pm\infty$:

$$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) := \left\{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} : f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}), \exists \lim_{t \rightarrow \pm\infty} f^{(j)}(t) \in \mathbb{R}, j = 0, \dots, n \right\},$$

being $\overline{\mathbb{R}} \equiv [-\infty, \infty]$. It holds that $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$, $n \in \mathbb{N}$ is a Banach space with the norm

$$\|f\|_{(n)} := \sup \left\{ \|f^{(k)}\|_{\infty} : k = 0, \dots, n \right\}.$$

Then, for a given function $\varphi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$, we define the space of continuously n -differentiable φ -extensions to infinity as follows:

$$\tilde{\mathcal{C}}_{\varphi}^n \equiv \tilde{\mathcal{C}}_{\varphi}^n(\mathbb{R}, \mathbb{R}) = \left\{ f \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}) : \exists \tilde{f} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}), f = \varphi \tilde{f}|_{\mathbb{R}} \right\}.$$

In particular, this is a Banach space with the induced norm

$$\|f\|_{\varphi} := \|\tilde{f}\|_{(n)}, \quad f \in \tilde{\mathcal{C}}_{\varphi},$$

from where it is deduced that the spaces $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ and $\tilde{\mathcal{C}}_{\varphi}^n$ are isomorphic.

From the existence of this isomorphism it is inferred that, since Ascoli-Arzelà's Theorem can be applied to the space $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ (since $\overline{\mathbb{R}}$ is compact), then this theorem is also applicable to $\tilde{\mathcal{C}}_{\varphi}^n$.

Then, we will look for fixed points of the integral operator

$$Tu(t) := p(t) + \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds$$

in the Banach space \tilde{C}_{φ}^n , for a given function φ which will precisely represent the asymptotic behavior of the solutions. In other words, the fact that the fixed points of the operator belong to the space \tilde{C}_{φ}^n will imply that such functions will asymptotically behave in a similar way to φ .

Regarding the method employed to guarantee the existence of fixed points, we consider two different approaches in this chapter: the first of them, developed in Section 8.4, is based on the fixed point index in cones and presents quite restrictive hypotheses for the non linearity f .

On the other hand, the second approach, considered in Section 8.5, is based on the definition and spectral properties of several auxiliary linear operators. In particular, if the spectral radius of these operators together with some limits involving the nonlinearity f satisfy some suitable properties, it will be possible to ensure the existence of fixed points. In this case, the restrictions on the function f are much weaker than the ones imposed with the previous method, but at the expense of requiring the kernel k to satisfy some more restrictive conditions. As it is shown in the chapter with some examples, the two methods are not comparable.

All these results are collected in [33] and [34].

Chapter 9: On Unbounded Solutions of Singular Initial Value Problems with ϕ -Laplacian

In this last chapter we study a singular initial value problem with ϕ -Laplacian, with special attention to the existence of unbounded solutions.

In this case, since we are dealing with a singular problem, it is not possible to construct an equivalent integral problem, as it was made in previous chapters. Consequently, the techniques used in this chapter will totally differ from the ones considered up to this moment.

In particular, we will consider the following nonlinear problem:

$$\begin{cases} (p(t) \phi(u'(t)))' + p(t) f(\phi(u(t))) = 0, & t > 0, \\ u(0) = u_0, \quad u'(0) = 0, & u_0 \in [L_0, L]. \end{cases}$$

We begin the chapter with the definition of three types of solutions that we may find. This way, denoting

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\},$$

we will say that

- A solution u of the problem is damped if $u_{\sup} < L$.
- A solution u of the problem is homoclinic if $u_{\sup} = L$.
- A solution u of the problem will be an escape solution if $u_{\sup} > L$.

Since both damped and homoclinic solutions are bounded, unbounded solutions will constitute a subset of the escape ones. This motivates the division of the chapter in two parts:

1. Search of conditions to ensure the existence of escape solutions.
2. Search of necessary and sufficient conditions to guarantee that an escape solution is unbounded.

Moreover, for the investigation of conditions which assure the existence of escape solutions, we will distinguish two differentiated cases: the first of them, in which both f and ϕ^{-1} are Lipschitz continuous, is quite simple since, under these conditions, we can guarantee the uniqueness of solution of the problem.

On the contrary, the second case (with f and ϕ^{-1} not necessarily Lipschitz continuous), presents several complications derived from the non uniqueness of solution. To solve these problems, we consider the method of lower and upper solutions.

These two cases have another important difference in relation with the results obtained: in the first one, we guarantee the existence of a sequence of escape solutions with different initial values, whereas in the second one, it may occur that all the solutions have the same initial value L_0 .

Finally, the last section of the chapter compiles all of the obtained results. We give there the explicit formulation of some sufficient conditions to assure the existence of unbounded solutions of the problem. Several examples show that all these results are not comparable.

All the results in this chapter are given in [131].



Part I

Bounded Domains



The first part of this Thesis focuses on the study of linear and nonlinear boundary value problems defined on bounded domains.

Our main goal will be the study of nonlinear differential equations. However, when working with this kind of problems, the first step consists of studying their related linear ones. To do this, the main tool will be the so-called Green's function, whose properties will clearly lead to the best way of dealing with nonlinear problems.

In particular, some of the main techniques applied in the recent literature to prove the existence of solutions of nonlinear boundary value problems are, among others, monotone iterative techniques (see [74, 96, 143]), the lower and upper solutions method (see [16, 46]) or fixed points theorems (see [74, 142]). In all these cases, the constant sign of the associated Green's functions is usually fundamental to prove such results.

All this clearly justifies the necessity of starting this Thesis by studying linear boundary value problems, focusing our attention on the properties which characterize the constant sign of Green's functions.

This part is structured in six chapters as follows:

Chapter 1 is dedicated to show some preliminary results and concepts for the sake of constructing a self-contained thesis. First, following [18], we define the Green's function related to a boundary value problem. Then, we summarize some results which will be used throughout this Thesis to ensure the existence of fixed points of various operators defined on Banach spaces. Finally, we include some definitions and results of spectral theory.

Chapter 2 includes a fully-detailed study of even order linear boundary value problems, focusing on finding relations between various Green's functions. All the results in this chapter are collected in [31].

Chapter 3 particularizes the study developed in Chapter 2 to the second order equation. Since Sturm-Liouville and Oscillation theory is applicable to second order problems, the results in this chapter are more powerful than those in the previous one. This chapter compiles results included in [22] and [23].

Chapter 4 deals, for the first time, with nonlinear boundary value problems. In particular, the results obtained in Chapters 2 and 3 will be shown to be of great importance to ensure the existence of solutions of the nonlinear problems considered in this chapter. A basic assumption will be the constant sign of the considered Green's functions. Moreover, the tool used to prove the existence of solution will be the lower and upper solutions method. This chapter is based on the last section of [31].

Chapter 5 is completely devoted to find solutions of nonlinear problems in the case when, contrary to Chapter 4, the Green's functions change sign. In particular, we will deal with second order problems and prove the existence of solutions by means of the fixed point index theory. This results are collected in [27].

Finally, Chapter 6 considers an integral problem instead of a differential one. As we will see, this is in fact a generalization since differential problems can be transformed into integral ones whose kernel function is, precisely, the Green's function. In this chapter, we will define a new type of cones which makes it necessary to apply the fixed point index theory in unbounded sets in order to find fixed points of the considered integral operators. Sections 6.1 to 6.6 of this chapter are included in [102], while the particular case given in Section 6.7 is collected in [32].



Chapter 1

Preliminaries

In order to construct a self-contained work, we will dedicate this chapter to introduce some definitions and previous results which will be used throughout the different chapters.

First of all, since the main tool to study linear problems is the so-called Green's function, Section 1.1 introduces this concept and establishes how the solutions of linear problems can be explicitly calculated by means of the Green's function.

Next, we will show in Section 1.2 how to transform a nonlinear differential problem into an equivalent integral one, in the sense that the solutions of the aforementioned differential problem correspond to fixed points of a certain integral operator. Moreover, we will include in this section several theorems to prove the existence of fixed points of compact operators defined on Banach spaces.

Finally, Section 1.3 compiles some basic results of spectral theory of linear operators.

1.1. Green's Functions

In this section, following [18], we will summarize the definition and main properties of Green's functions.

Consider the general two-point n -th order differential problem

$$\begin{cases} L_n u(t) = \sigma(t), & t \in [0, T], \\ U_i(u) = 0, & i = 1, \dots, n, \end{cases} \quad (1.1.1)$$

where

$$L_n u(t) \equiv u^{(n)}(t) + a_{n-1}(t) u^{(n-1)}(t) + \dots + a_1(t) u'(t) + a_0(t) u(t)$$

and

$$U_i(u) \equiv \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(0) + \beta_j^i u^{(j)}(T) \right), \quad i = 1, \dots, n,$$

where α_j^i and β_j^i are real constants for all $i = 1, \dots, n$, $j = 0, \dots, n-1$ and $\sigma, a_k \in L^1([0, T])$ for all $k = 1, \dots, n$.

We will denote $I \equiv [0, T]$.

We can characterize the Green's function for problem (1.1.1) as follows.

Definition 1.1.1. [18, Definition 1.4.1] We say that G is a Green's function for problem (1.1.1) if it satisfies the following properties:

(G1) G is defined on the square $I \times I$ (except at the points with $t = s$ if $n = 1$).

(G2) For $k = 0, \dots, n-2$, the partial derivatives $\frac{\partial^k G}{\partial t^k}$ exist and are continuous on $I \times I$.

(G3) Both $\frac{\partial^{n-1} G}{\partial t^{n-1}}$ and $\frac{\partial^n G}{\partial t^n}$ exist and are continuous on the triangles $0 \leq s < t \leq T$ and $0 \leq t < s \leq T$.

(G4) For each $s \in (0, T)$, the function $G(\cdot, s)$ is a solution of the differential equation $L_n y = 0$ a. e. on $[0, s) \cup (s, T]$, that is,

$$\frac{\partial^n G}{\partial t^n}(t, s) + a_{n-1}(t) \frac{\partial^{n-1} G}{\partial t^{n-1}}(t, s) + \dots + a_1(t) \frac{\partial G}{\partial t}(t, s) + a_0(t) G(t, s) = 0,$$

for all $t \in I \setminus \{s\}$.

(G5) For each $t \in (0, T)$ there exist the lateral limits

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(t^-, t) = \frac{\partial^{n-1} G}{\partial t^{n-1}}(t, t^+) \text{ and } \frac{\partial^{n-1} G}{\partial t^{n-1}}(t, t^-) = \frac{\partial^{n-1} G}{\partial t^{n-1}}(t^+, t)$$

and, moreover,

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(t^+, t) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(t^-, t) = \frac{\partial^{n-1} G}{\partial t^{n-1}}(t, t^-) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(t, t^+) = 1.$$

(G6) For each $s \in (0, T)$, the function $G(\cdot, s)$ satisfies the boundary conditions $U_i(G(\cdot, s)) = 0$, $i = 1, \dots, n$, that is,

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j G}{\partial t^j}(0, s) + \beta_j^i \frac{\partial^j G}{\partial t^j}(T, s) \right) = 0, \quad i = 1, \dots, n.$$

Remark 1.1.2. Note that the Green's function depends on the homogeneous part of problem (1.1.1), but not on the considered function σ . Due to this fact, we will frequently talk about the Green's function related to the homogeneous problem, namely

$$\begin{cases} L_n u(t) = 0, & t \in [0, T], \\ U_i(u) = 0, & i = 1, \dots, n. \end{cases}$$

We will consider the space

$$W^{n,1}(I) = \left\{ u \in C^{n-1}(I) : u^{(n-1)} \in \mathcal{AC}(I) \right\},$$

where $\mathcal{AC}(I)$ denotes the set of absolutely continuous functions on I .

In particular, we will consider a subset $X \subset W^{n,1}(I)$ defined in the following way

$$X = \left\{ u \in W^{n,1}(I) : U_i(u) = 0, i = 1, \dots, n \right\}. \quad (1.1.2)$$

It is easy to check that X is a Banach space with the usual norm

$$\|u\|_X = \max \left\{ \|u^{(i)}\| : i = 0, \dots, n-1 \right\}.$$

Now, we will introduce the following definition.

Definition 1.1.3. *Given a Banach space X , operator L_n is said to be nonresonant on X if and only if the homogeneous equation*

$$L_n u(t) = 0 \quad \text{a. e. } t \in I, \quad u \in X,$$

has only the trivial solution.

The following result relates the uniqueness of solution of problem (1.1.1) with the uniqueness of the Green's function. This can be seen in [18, Corollary 1.2.4 and Theorem 1.2.17].

Theorem 1.1.4. *The following assertions are equivalent:*

1. *Operator L_n is nonresonant on X given in (1.1.2).*
2. *There exists a unique Green's function related to problem (1.1.1).*
3. *Problem (1.1.1) has a unique solution $u \in W^{n,1}(I)$.*

In such a case, the unique solution is given by the following expression

$$u(t) = \int_0^T G(t, s) \sigma(s) \, ds, \quad \forall t \in I. \quad (1.1.3)$$

Furthermore, it is very well known (see [18, 40, 106]) that operator L_n is self-adjoint on X if and only if the related Green's function exists and is symmetrical with respect to the diagonal of its square of definition, that is,

$$G(t, s) = G(s, t), \quad \forall (t, s) \in I \times I.$$

We will also introduce the following important definitions. In them, we will use the notation $h \succ 0$ to denote a function $h \in L^\alpha(I)$ such that $h(t) \geq 0$ for a. e. $t \in I$ and $h \not\equiv 0$ on I .

Definition 1.1.5. Operator L_n is said to admit the maximum principle (MP) on X if and only if every function $u \in X$ such that $L_n u \succ 0$ on I satisfies that $u < 0$ on $(0, T)$.

Definition 1.1.6. Operator L_n is said to admit the antimaximum principle (AMP) on X if and only if every function $u \in X$ such that $L_n u \succ 0$ on I satisfies that $u > 0$ on $(0, T)$.

It is immediate to verify that if L_n satisfies MP or AMP on X then it is nonresonant on X .

Moreover, if L_n is self-adjoint, the previously defined maximum and antimaximum principles can be related with the constant sign of the Green's function. Next result was first proved in [160, Theorem 4.1] for the second order equation and the Green's function related to the periodic problem. The proof for second order equation and arbitrary boundary conditions can be found in [23, Lemma 10]. For the reader's convenience, we include now the proof for any arbitrary order.

Theorem 1.1.7. If L_n is a self-adjoint operator, then the following equivalences hold:

- Operator L_n satisfies MP on X if and only if the related Green's function is nonpositive on $I \times I$.
- Operator L_n satisfies AMP on X if and only if the related Green's function is nonnegative on $I \times I$.

Proof. First we will prove that if operator L_n satisfies one of the principles, then the related Green's function has constant sign.

Assume, on the contrary, that operator L_n satisfies either MP or AMP and suppose that G changes sign on $I \times I$. Arguing as in [16, Theorem 3.1], one can find $t_0 \in I$ and $u_1, u_2 \in X$ such that $L_n u_1 \succ 0$, $L_n u_2 \succ 0$ on I and $u_1(t_0) u_2(t_0) < 0$.

First we will prove that there exist $t_0 \in (0, T)$, $s_1, s_2 \in I$ such that $G(t_0, s_1) > 0$ and $G(t_0, s_2) < 0$. On the contrary, if $G(t, \cdot)$ has constant sign for all $t \in (0, T)$ then, due to the change of sign of G , there will exist some $t_1 \in (0, T)$ such that $G(t_1, \cdot) \equiv 0$. From symmetry, $G(\cdot, t_1) \equiv 0$, which contradicts the fact that

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(t_1^+, t_1) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(t_1^-, t_1) = 1.$$

Therefore the existence of such t_0 is ensured.

Then there is a neighbourhood of s_1 , $A_1 \subset [0, T]$, in which $G(t_0, \cdot)$ is positive. If we choose a function f_1 which is positive on A_1 and vanishes on $[0, T] \setminus A_1$, we have that there exists u_1 satisfying that $L_n u_1 = f_1 \succ 0$ and

$$u_1(t_0) = \int_{A_1} G(t_0, s) f_1(s) \, ds > 0.$$

Analogously, there is a neighbourhood of s_2 , $A_2 \subset [0, T]$, in which $G(t_0, \cdot)$ is negative. Choosing now f_2 such that it is positive on A_2 and vanishes on $[0, T] \setminus A_2$, we have that there exists u_2 such that $L_n u_2 = f_2 \succ 0$ and

$$u_2(t_0) = \int_{A_2} G(t_0, s) f_2(s) \, ds < 0.$$

Therefore, we reach a contradiction.

We will see now the reciprocal, that is, we will prove that the constant sign of the Green's function implies one of the principles.

First observe that inequality $L_n u \succ 0$ on I is equivalent to the existence of some $\sigma \in L^1(I)$ such that $\sigma \succ 0$ on I , for which

$$L_n u(t) = \sigma(t), \quad t \in I.$$

Then, if the Green's function does not change sign, we deduce the strict constant sign of u on $(0, T)$ as a direct consequence of (1.1.3) and the fact that, as we have just seen, it can not exist any $t \in (0, T)$ for which $G(t, \cdot) \equiv 0$. \square

To finish with this preliminary subsection, we will show two particular cases of some more general spectral results given in [18, Lemmas 1.8.25 and 1.8.33]. For these results we need to introduce a new differential operator.

For any $\lambda \in \mathbb{R}$, consider operator $L_n[\lambda]$ defined from operator L_n in the following way

$$L_n[\lambda] u(t) \equiv u^{(n)}(t) + a_{n-1}(t) u^{(n-1)}(t) + \cdots + a_1(t) u'(t) + (a_0(t) + \lambda) u(t), \quad t \in I,$$

that is, $L_n[\lambda] u(t) = L_n u(t) + \lambda u(t)$.

To stress its dependence on λ , we will denote by $G[\lambda]$ the Green's function related to $L_n[\lambda]$.

We have the following results.

Lemma 1.1.8. *Suppose that operator L_n is nonresonant on a Banach space X , its related Green's function G is nonpositive on $I \times I$, and satisfies the following condition.*

(N_g) *There is a continuous function $\phi(t) > 0$ for all $t \in (0, T)$ and $k_1, k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a. e. $s \in I$, satisfying*

$$\phi(t) k_1(s) \leq G(t, s) \leq \phi(t) k_2(s), \quad \text{for a. e. } (t, s) \in I \times I.$$

Then $G[\lambda]$ is nonpositive on $I \times I$ if and only if $\lambda \in (-\infty, \lambda_1)$ or $\lambda \in [-\bar{\mu}, \lambda_1)$, with $\lambda_1 > 0$ the first eigenvalue of operator L_n in X and $\bar{\mu} \geq 0$ such that $L_n[-\bar{\mu}]$ is nonresonant on X and the related nonpositive Green's function $G[-\bar{\mu}]$ vanishes at some point of the square $I \times I$.

Lemma 1.1.9. *Suppose that operator L_n is nonresonant on a Banach space X , its related Green's function G is nonnegative on $I \times I$, and satisfies the following condition.*

(P_g) *There is a continuous function $\phi(t) > 0$ for all $t \in (0, T)$ and $k_1, k_2 \in L^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a. e. $s \in I$, satisfying*

$$\phi(t) k_1(s) \leq G(t, s) \leq \phi(t) k_2(s), \quad \text{for a. e. } (t, s) \in I \times I.$$

Then $G[\lambda]$ is nonnegative on $I \times I$ if and only if $\lambda \in (\lambda_1, \infty)$ or $\lambda \in (\lambda_1, \bar{\mu}]$, with $\lambda_1 < 0$ the first eigenvalue of operator L_n in X and $\bar{\mu} \geq 0$ such that $L_n[\bar{\mu}]$ is nonresonant on X and the related nonnegative Green's function $G[\bar{\mu}]$ vanishes at some point of the square $I \times I$.

It is obvious that if the Green's function is strictly positive (respectively, strictly negative) on $I \times I$ then condition (P_g) (respectively, (N_g)) is trivially fulfilled.

1.2. Fixed Point Theorems

This section is devoted to present some sufficient conditions that ensure the existence of fixed points of operators defined in abstract spaces.

As it has been indicated before, Chapters 4 to 9 will be devoted to find conditions to ensure the existence of solution of nonlinear problems. In this framework, as we will show now, fixed point theorems will constitute a basic tool to deal with the aforementioned problems.

Consider for instance the following nonlinear boundary value problem related to problem (1.1.1)

$$\begin{cases} L_n u(t) = f(t, u(t)), & t \in I, \quad u \in X \\ U_i(u) = 0, & i = 1, \dots, n, \end{cases}$$

with $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying some suitable regularity conditions (which, as we will see, will actually vary depending on the problem).

It is a very well-known result that the solutions of previous problem coincide with the fixed points of the following integral operator

$$\begin{aligned} \mathcal{T}: X &\longrightarrow X \\ u &\longmapsto \mathcal{T} u \end{aligned}$$

where

$$\mathcal{T} u(t) = \int_0^T G(t, s) f(s, u(s)) \, ds,$$

with G the Green's function related to problem (1.1.1).

It is now when results ensuring the existence of fixed points of operators in Banach spaces arise as the key to deal with nonlinear problems. We compile in this section some of those results that will be used throughout this Thesis.

First, we will introduce the concept of compact operator as follows (note that in some references these operators are named as completely continuous).

Definition 1.2.1 ([101, Definition 4.2.1]). *Given two Banach spaces X and Y , we say that an operator $\mathcal{T} : X \rightarrow Y$ is compact if and only if it satisfies the two following properties:*

- \mathcal{T} is continuous.
- \mathcal{T} maps bounded subsets of X into relatively compact subsets of Y .

A very useful tool to prove the compactness of an operator is the very well-known Ascoli-Arzelà's Theorem.

Theorem 1.2.2 (Ascoli-Arzelà, [89, Chapter 7, Theorem 18]). *Let X be a Hausdorff compact topological space and Y a complete metric space, and consider $\mathcal{C}(X, Y)$ with the topology of the uniform convergence. Then $F \subset \mathcal{C}(X, Y)$ has compact closure if and only if the two following properties hold:*

- $F(x)$ has compact closure for each $x \in X$.
- F is equicontinuous.

The following theorem was proved by Schauder in [132].

Theorem 1.2.3 (Schauder). *Let S be a bounded, closed, nonempty, convex subset of the normed space X . Let \mathcal{T} be a compact operator such that $\mathcal{T}(S) \subset S$. Then \mathcal{T} has a fixed point in S .*

On the other hand, a very useful tool to ensure the existence of fixed points of compact operators is the fixed point index theory in cones.

Definition 1.2.4. *Given a Banach space X , we say that $K \subset X$ is a cone if it is a closed and convex subset of X satisfying the two following properties:*

1. *If $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$.*
2. *$K \cap (-K) = \{0\}$.*

A cone K induces a partial order in X in the following way: $x \preceq y$ if and only if $y - x \in K$.

Definition 1.2.5. We say that K is a total cone if $\overline{K - K} = X$.

We will compile now some classical results regarding the fixed point index. In particular, we will follow the line of [64, Chapter 12]. There, this theory is developed in a more general framework, namely, for the so-called absolute neighbourhood retracts. However, since throughout this Thesis we will work with Banach spaces (and, as it is stated in [64, Chapter 11, Corollary 5.4], a Banach space is a particular case of absolute neighbourhood retract), we will reformulate all the results in terms of Banach spaces.

Therefore, from now on, we will assume that X is a Banach space, $\Omega \subset X$ an arbitrary open subset and \mathcal{T} a compact operator.

In [64, Chapter 12], the fixed point index is defined for compact maps which are compactly fixed.

Definition 1.2.6. Let X be a space, $\Omega \subset X$ open and $\mathcal{T}: U \rightarrow X$ a continuous map. We say that \mathcal{T} is compactly fixed if the set of fixed points of \mathcal{T} is compact.

We will denote by $\text{Fix}(\mathcal{T})$ the set of fixed points of \mathcal{T} .

Next lemma compiles some classical results regarding the fixed point index formulated in [64, Theorems 6.2, 7.3 and 7.11] in a more general framework.

In particular, given X a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an arbitrary open subset, $\partial\Omega$ will denote the boundary of Ω in the relative topology in K , induced by the topology of X .

Lemma 1.2.7. Let X be a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an arbitrary open subset with $0 \in \Omega$. Assume that $\mathcal{T}: \overline{\Omega} \rightarrow K$ is a compact and compactly fixed operator such that $x \neq \mathcal{T}x$ for all $x \in \partial\Omega$.

Then the fixed point index $i_K(\mathcal{T}, \Omega)$ has the following properties:

1. If $x \neq \mu \mathcal{T}x$ for all $x \in \partial\Omega$ and for every $\mu \leq 1$, then $i_K(\mathcal{T}, \Omega) = 1$.
2. If Ω is bounded and there exists $e \in K \setminus \{0\}$ such that $x \neq \mathcal{T}x + \lambda e$ for all $x \in \partial\Omega$ and all $\lambda > 0$, then $i_K(\mathcal{T}, \Omega) = 0$.
3. If $i_K(\mathcal{T}, \Omega) \neq 0$, then \mathcal{T} has a fixed point in Ω .
4. If Ω_1 and Ω_2 are two open and disjoint sets such that $\text{Fix}(\mathcal{T}) \subset \Omega_1 \cup \Omega_2 \subset \Omega$, then

$$i_K(\mathcal{T}, \Omega) = i_K(\mathcal{T}, \Omega_1) + i_K(\mathcal{T}, \Omega_2).$$

Remark 1.2.8. Note that, in Item 2 in previous lemma, it is required that Ω is bounded. However, the other assertions hold for an arbitrary open set, which might be unbounded.

The following result gives more sufficient conditions to ensure that the index of an arbitrary open subset is 1.

Corollary 1.2.9 ([64, Corollary 7.4]). *Let X be a Banach space with the norm $\|\cdot\|$, $K \subset X$ a cone and $\Omega \subset K$ an arbitrary open subset with $0 \in \Omega$. Moreover, let $\mathcal{T}: \overline{\Omega} \rightarrow K$ be a compact and compactly fixed operator without fixed points in $\partial\Omega$. Assume that one of the following conditions holds for all $x \in \partial\Omega$:*

- (i) $\|\mathcal{T}x\| \leq \|x\|$.
- (ii) $\|\mathcal{T}x\| \leq \|x - \mathcal{T}x\|$.
- (iii) $\|\mathcal{T}x\|^2 \leq \|x\|^2 + \|x - \mathcal{T}x\|^2$.
- (iv) $\langle x, \mathcal{T}x \rangle \leq \langle x, x \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product in X .

Then $i_K(\mathcal{T}, \Omega_K) = 1$.

Moreover, using Items 1 and 2 in Lemma 1.2.7, it is possible to deduce the following corollary. The proof would be analogous to that of [68, Theorem 2.3.3].

Corollary 1.2.10. *Let X be a Banach space, $K \subset X$ a cone and $\Omega \subset K$ an open set such that $0 \in \Omega$. Assume that $\mathcal{T}: \overline{\Omega} \rightarrow K$ is a compact and compactly fixed operator such that $x \neq \mathcal{T}x$ for all $x \in \partial\Omega$. Then*

1. *If $\mathcal{T}x \not\leq x$ for all $x \in \partial\Omega$ then $i_K(\mathcal{T}, \Omega) = 1$.*
2. *If Ω is bounded and, moreover, $\mathcal{T}x \not\leq x$ for all $x \in \partial\Omega$, then $i_K(\mathcal{T}, \Omega) = 0$.*

1.3. Spectral Theory

Another important tool when working with linear operators is given by the spectral theory.

As we will see, the existence of a positive eigenfunction will be a basic assumption in some results in Chapters 2 and 3, and one of the most powerful results to ensure this is the very well-known Krein-Rutman Theorem.

Moreover, as we will show in Chapter 8, it is also possible to prove the existence of fixed points of nonlinear integral operators by studying spectral properties of some related linear operators.

We will compile now some results regarding spectral theory of linear operators defined on normed spaces, which we will use in the following chapters.

Let $(N_1, \|\cdot\|_1)$ and $(N_2, \|\cdot\|_2)$ be two normed spaces. Let $\mathcal{T} : N_1 \rightarrow N_2$ be a bounded linear operator, that is, such that its norm

$$\|\mathcal{T}\| = \sup_{\|u\|_2 \neq 0} \frac{\|\mathcal{T}u\|_1}{\|u\|_2}$$

is finite. We recall the following definitions.

Definition 1.3.1. We say that λ is an eigenvalue of a linear operator between normed spaces $\mathcal{T} : (N_1, \|\cdot\|_1) \rightarrow (N_2, \|\cdot\|_2)$, with corresponding eigenfunction ϕ , if $\phi \neq 0$ and $\lambda \phi = \mathcal{T}\phi$.

The reciprocals of nonzero eigenvalues are called characteristic values of \mathcal{T} .

Definition 1.3.2. We will define the spectral radius of a bounded linear operator \mathcal{T} as

$$r(\mathcal{T}) := \lim_{n \rightarrow \infty} \|\mathcal{T}^n\|^{\frac{1}{n}},$$

and its principal characteristic value as $\mu(\mathcal{T}) := \frac{1}{r(\mathcal{T})}$ if $r(\mathcal{T}) \neq 0$.

For more properties of this generalized spectral value we refer the reader to [11, 164].

Now we will formulate the very well-known Krein-Rutman Theorem.

Theorem 1.3.3 (Krein-Rutman, [52, Theorem 1.1]). Let $K \subset X$ be a total cone and $\mathcal{T} : X \rightarrow X$ a compact linear operator that maps K to K with positive spectral radius $r(\mathcal{T})$. Then $r(\mathcal{T})$ is an eigenvalue with an eigenvector $\phi \in K \setminus \{0\}$.

We will give now the sharper version of this theorem for strongly positive linear operators.

Definition 1.3.4. Let $K \subset X$ be a cone with nonempty interior and let $\mathcal{T} : X \rightarrow X$ be a compact linear operator. We will say that \mathcal{T} is strongly positive if and only if

$$\mathcal{T}x \in \text{int}(K), \quad \forall x \in K \setminus \{0\},$$

where $\text{int}(K)$ denotes the interior of the cone.

Theorem 1.3.5 ([4, Theorem 3.2]). Let $K \subset X$ be a cone with nonempty interior and $\mathcal{T} : X \rightarrow X$ a strongly positive and compact linear operator that maps K to K . Then, the following assertions hold:

- The spectral radius $r(\mathcal{T})$ is positive.
- $r(\mathcal{T})$ is a simple eigenvalue of \mathcal{T} with a positive related eigenfunction and there is no other eigenvalue with a positive eigenfunction.

Finally, we recall some known results which will let us find some lower and upper bounds for the spectral radius.

Theorem 1.3.6 ([148, Theorem 2.7]). *Let \mathcal{T} be a bounded linear operator in a Banach space X and let K be a cone in X such that $\mathcal{T}(K) \subset K$. If there exists $\lambda_0 > 0$ and $v \in K \setminus \{0\}$ such that*

$$\mathcal{T}v \succeq \lambda_0 v,$$

then $r(\mathcal{T}) \geq \lambda_0$.

Theorem 1.3.7 ([157, Theorem 1]). *Let \mathcal{T} be a linear and compact operator and let K be a cone in X . Assume that K has non empty interior and that $\mathcal{T}(K) \subset K$. If there exists v , an interior element of the cone, for which the following inequality holds*

$$\mathcal{T}v \preceq \lambda_0 v,$$

then $r(\mathcal{T}) \leq \lambda_0$.





Green's Functions and Spectral Theory for Even Order Linear Boundary Value Problems

In this chapter we will develop a fully-detailed study of even order linear boundary value problems.

We have seen in the previous chapter that the solutions of a given boundary value problem coincide with the fixed points of related integral operators which have as kernel the associated Green's function in each case. Thus, the Green's function plays a very important role in the study of boundary value problems.

Traditionally, the most studied boundary value problems have been the periodic and the two-point ones. In this chapter we will take advantage of such studies by finding some connections between the Green's functions of various separated two point boundary conditions and the Green's functions of periodic problem. The key idea is that the expression of the Green's function related to each two points case can be obtained as a linear combination of the Green's function of periodic problems.

From these expressions relating the different Green's functions, we will be able to compare their constant sign.

These results will allow us to obtain some comparison principles which guarantee that, under certain hypotheses, the solution of a boundary value problem under some suitable conditions is bigger in every point than the solution of the same equation under another type of boundary conditions.

We will also obtain a decomposition of the spectrum of some problems as a combination of the other ones and some relations of order between the first eigenvalues of the considered problems.

The chapter is organized as follows: Section 2.1 includes some preliminary results and proves a symmetry property which will be satisfied by some Green's functions. In Section 2.2, we detail the aforementioned decomposition of Green's functions. In Section 2.3, we relate both the spectra and the first eigenvalues of the considered problems. In Section 2.4, we prove some results relating the constant sign of various Green's functions. Finally, in Section 2.5, we show some point-by-point

relations between different Green's functions and also between solutions of the same operator under several boundary conditions.

It must be pointed out that the study developed in Sections 2.3 to 2.5 will be particularized in Chapter 3 for the second order equation. As we will see, in such a case, many results will be stronger than for the general even order problem. The reason is that, for second order equations, Sturm-Liouville's Theory is applicable, which makes it possible to obtain more information regarding oscillation of the solutions, Green's functions and spectral theory.

All the results in this chapter are collected in [31].

2.1. Preliminary Results

In this section we will introduce three different operators. The first of them, which will be called operator L , will be defined with arbitrary coefficients. On the other hand, the coefficients of the other two operators (denoted by \tilde{L} and $\tilde{\tilde{L}}$) will be defined as either even or odd extensions of the coefficients of the aforementioned operator L . This way, while the original operator will be defined on the interval $[0, T]$, the two auxiliary operators \tilde{L} and $\tilde{\tilde{L}}$ will be defined on $[0, 2T]$ and $[0, 4T]$, respectively.

Furthermore, the symmetries in the coefficients of the operator \tilde{L} will induce also some symmetries on the Green's functions related to this operator. This property will also be proved in this section.

Consider then the $2n$ -th order general linear operator

$$L u(t) \equiv u^{(2n)}(t) + a_{2n-1}(t) u^{(2n-1)}(t) + \cdots + a_1(t) u'(t) + a_0(t) u(t), \quad (2.1.1)$$

with $t \in I$ and $a_k : I \rightarrow \mathbb{R}$, $a_k \in L^\alpha(I)$, $\alpha \geq 1$, $k = 0, \dots, 2n-1$.

We will introduce now the first auxiliary linear operator, whose coefficients will be defined from those of operator L as follows:

$$\begin{aligned} \tilde{L} u(t) \equiv & u^{(2n)}(t) + \hat{a}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{a}_{2n-2}(t) u^{(2n-2)}(t) \\ & + \cdots + \hat{a}_1(t) u'(t) + \tilde{a}_0(t) u(t), \quad t \in J \equiv [0, 2T], \end{aligned}$$

where \tilde{a}_{2k} , $k = 0, \dots, n-1$, is the even extension of a_{2k} to J , that is,

$$\tilde{a}_{2k}(t) = \begin{cases} a_{2k}(t), & t \in [0, T], \\ a_{2k}(2T - t), & t \in [T, 2T], \end{cases}$$

and \hat{a}_{2k+1} , $k = 0, \dots, n-1$, is the odd extension of a_{2k+1} to J , that is,

$$\hat{a}_{2k+1}(t) = \begin{cases} a_{2k+1}(t), & t \in [0, T], \\ -a_{2k+1}(2T - t), & t \in (T, 2T]. \end{cases}$$

Notation 2.1.1. As we have mentioned before, throughout this chapter we will work with problems defined on different intervals. Because of this reason, we will use the notation $G[T]$, $G[2T]$ and $G[4T]$ to indicate that we are working on the interval $[0, T]$, $[0, 2T]$ or $[0, 4T]$, respectively. This way, we will stress the dependence of the Green's function on the considered interval.

We obtain the following symmetric property for Green's functions related to operator \tilde{L} .

Lemma 2.1.2. Let $X \subset W^{2n,1}(J)$ be a Banach space such that operator \tilde{L} is nonresonant on X . Moreover, suppose that if $v \in X$ and $w \in W^{2n,1}(J)$ is such that $w(t) := v(2T - t)$ for all $t \in J$, then $w \in X$. Then the following equality holds:

$$G[2T](t, s) = G[2T](2T - t, 2T - s), \quad \forall (t, s) \in J \times J. \quad (2.1.2)$$

Proof. Let $\bar{\sigma} \in L^1(J)$ be arbitrarily chosen and consider the problem

$$\tilde{L}v(t) = \bar{\sigma}(t), \quad \text{a.e. } t \in J, \quad v \in X.$$

Since operator \tilde{L} is nonresonant on X , this problem has a unique solution v which is given by

$$v(t) = \int_0^{2T} G[2T](t, s) \bar{\sigma}(s) \, ds.$$

On the other hand, taking into account the fact that $\tilde{a}_{2k}(t) = \tilde{a}_{2k}(2T - t)$ and $\hat{a}_{2k+1}(t) = -\hat{a}_{2k+1}(2T - t)$, it is easy to verify that $w(t) = v(2T - t)$ is the unique solution of the problem

$$\tilde{L}w(t) = \bar{\sigma}(2T - t), \quad \text{a.e. } t \in J, \quad w \in X.$$

Therefore,

$$w(t) = \int_0^{2T} G[2T](t, s) \bar{\sigma}(2T - s) \, ds$$

and, making a suitable change of variable,

$$w(t) = \int_0^{2T} G[2T](t, 2T - s) \bar{\sigma}(s) \, ds.$$

Now, since

$$w(t) = v(2T - t) = \int_0^{2T} G[2T](2T - t, s) \bar{\sigma}(s) \, ds,$$

and $\bar{\sigma} \in L^1(J)$ is arbitrary, we arrive at the following equality

$$G[2T](2T - t, s) = G[2T](t, 2T - s), \quad \forall (t, s) \in J \times J,$$

or, which is the same,

$$G[2T](t, s) = G[2T](2T - t, 2T - s), \quad \forall (t, s) \in J \times J.$$

□

In addition, we will consider another auxiliary operator $\tilde{\tilde{L}}$ which will be constructed from \tilde{L} in the same way than \tilde{L} has been constructed from L , that is:

$$\begin{aligned} \tilde{\tilde{L}} u(t) \equiv & u^{(2n)}(t) + \hat{a}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{a}_{2n-2}(t) u^{(2n-2)}(t) \\ & + \cdots + \hat{a}_1(t) u'(t) + \tilde{a}_0(t) u(t), \quad t \in [0, 4T], \end{aligned}$$

where \tilde{a}_{2k} and \hat{a}_{2k+1} , $k = 0, \dots, n-1$, are the even and odd extensions to the interval $[0, 4T]$ of \tilde{a}_{2k} and \hat{a}_{2k+1} , respectively.

2.2. Decomposing Green's Functions

In this section we will obtain the expression of the Green's function of different two point boundary value problems (Neumann, Dirichlet and Mixed problems) as a sum of Green's functions of other related problems.

This decomposition has been detailed in [22] for the particular case of $n = 1$ and $a_1 \equiv 0$ and generalized in [31] for the general case with arbitrary n .

In particular, we will work with some problems related to operator L (and, consequently, defined on the interval $[0, T]$), some others related to operator \tilde{L} (and, consequently, defined on $[0, 2T]$) and the periodic problem related to $\tilde{\tilde{L}}$ (defined on $[0, 4T]$). In the sequel, we describe the different problems and boundary conditions we are dealing with:

- Neumann problem on the interval I :

$$\begin{cases} L u(t) = \sigma(t), & \text{a. e. } t \in I, \\ u^{(2k+1)}(0) = u^{(2k+1)}(T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (N, T)$$

- Dirichlet problem on the interval I :

$$\begin{cases} L u(t) = \sigma(t), & \text{a. e. } t \in I, \\ u^{(2k)}(0) = u^{(2k)}(T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (D, T)$$

- Mixed problem 1 on the interval I :

$$\begin{cases} L u(t) = \sigma(t), & \text{a. e. } t \in I, \\ u^{(2k+1)}(0) = u^{(2k)}(T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (M_1, T)$$

- Mixed problem 2 on the interval I :

$$\begin{cases} L u(t) = \sigma(t), & \text{a. e. } t \in I, \\ u^{(2k)}(0) = u^{(2k+1)}(T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (M_2, T)$$

- Periodic problem on the interval J :

$$\begin{cases} \tilde{L} u(t) = \bar{\sigma}(t), & \text{a. e. } t \in J, \\ u^{(k)}(0) = u^{(k)}(2T), & k = 0, \dots, 2n-1. \end{cases} \quad (P, 2T)$$

- Antiperiodic problem on the interval J :

$$\begin{cases} \tilde{L} u(t) = \bar{\sigma}(t), & \text{a. e. } t \in J, \\ u^{(k)}(0) = -u^{(k)}(2T), & k = 0, \dots, 2n-1. \end{cases} \quad (A, 2T)$$

- Neumann problem on the interval J :

$$\begin{cases} \tilde{L} u(t) = \bar{\sigma}(t), & \text{a. e. } t \in J, \\ u^{(2k+1)}(0) = u^{(2k+1)}(2T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (N, 2T)$$

- Dirichlet problem on the interval J :

$$\begin{cases} \tilde{L} u(t) = \bar{\sigma}(t), & \text{a. e. } t \in J, \\ u^{(2k)}(0) = u^{(2k)}(2T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (D, 2T)$$

- Periodic problem on the interval $[0, 4T]$:

$$\begin{cases} \tilde{\tilde{L}} u(t) = \bar{\bar{\sigma}}(t), & \text{a. e. } t \in [0, 4T], \\ u^{(k)}(0) = u^{(k)}(4T), & k = 0, \dots, 2n-1. \end{cases} \quad (P, 4T)$$

Now, we will show how to relate the expressions of different Green's functions.

We will assume that all the considered operators are nonresonant on the corresponding Banach space with suitable boundary conditions. Later, we will see in Section 2.3 that the aforementioned nonresonant character of all the operators is, in some sense, equivalent.

2.2.1. Neumann Problem

To begin with, we will decompose the Green's function related to problem (N, T) as sum of the Green's function related to $(P, 2T)$ evaluated in the same point and of the same function evaluated in another point which satisfies a symmetric relation.

First, suppose that operator L is nonresonant on the space

$$X_{N,T} = \left\{ u \in W^{2n,1}(I) : u^{(2k+1)}(0) = u^{(2k+1)}(T) = 0, k = 0, \dots, n-1 \right\},$$

that is, problem (N, T) has a unique solution in $W^{2n,1}(I)$ for all $\sigma \in L^1(I)$.

Moreover, assume that \tilde{L} is nonresonant on

$$X_{P,2T} = \left\{ u \in W^{2n,1}(J) : u^{(k)}(0) = u^{(k)}(2T), k = 0, \dots, 2n-1 \right\},$$

that is, problem $(P, 2T)$ has a unique solution in $W^{2n,1}(J)$ for all $\bar{\sigma} \in L^1(J)$.

Let u be the unique solution of problem (N, T) . Then, defining v as the even extension of u , it can be proved that $v \in W^{2n,1}(J)$ satisfies the equation $\tilde{L}v(t) = \bar{\sigma}(t)$ for the particular case of taking $\bar{\sigma}$ as the even extension of σ . Indeed, for $t \in [0, T]$, it holds that

$$\tilde{L}v(t) = Lu(t) = \sigma(t) = \bar{\sigma}(t)$$

and, for $t \in [T, 2T]$,

$$\begin{aligned} \tilde{L}v(t) &= v^{(2n)}(t) + \hat{a}_{2n-1}(t)v^{(2n-1)}(t) + \tilde{a}_{2n-2}(t)v^{(2n-2)}(t) \\ &\quad + \dots + \hat{a}_1(t)v'(t) + \tilde{a}_0(t)v(t) \\ &= u^{(2n)}(2T-t) - \hat{a}_{2n-1}(t)u^{(2n-1)}(2T-t) + \tilde{a}_{2n-2}(t)u^{(2n-2)}(2T-t) \\ &\quad + \dots - \hat{a}_1(t)u'(2T-t) + \tilde{a}_0(t)u(2T-t) \\ &= u^{(2n)}(2T-t) + \hat{a}_{2n-1}(2T-t)u^{(2n-1)}(2T-t) \\ &\quad + \tilde{a}_{2n-2}(2T-t)u^{(2n-2)}(2T-t) \\ &\quad + \dots + \hat{a}_1(2T-t)u'(2T-t) + \tilde{a}_0(2T-t)u(2T-t) \\ &= \bar{\sigma}(2T-t) = \bar{\sigma}(t). \end{aligned}$$

Moreover, it is clear that $v \in X_{P,2T}$ and thus v is a solution of problem $(P, 2T)$.

Therefore, if we denote by $G_N[T]$ and $G_P[2T]$ the Green's functions related to problems (N, T) and $(P, 2T)$, respectively, we obtain the following equalities for

$t \in I$:

$$\begin{aligned}
 \int_0^T G_N[T](t, s) \sigma(s) \, ds &= u(t) = v(t) = \int_0^{2T} G_P[2T](t, s) \bar{\sigma}(s) \, ds \\
 &= \int_0^T G_P[2T](t, s) \sigma(s) \, ds \\
 &\quad + \int_T^{2T} G_P[2T](t, s) \sigma(2T - s) \, ds \\
 &= \int_0^T (G_P[2T](t, s) + G_P[2T](t, 2T - s)) \sigma(s) \, ds.
 \end{aligned}$$

Now, since previous equality holds for every $\sigma \in L^1(I)$, we can deduce that

$$G_N[T](t, s) = G_P[2T](t, s) + G_P[2T](t, 2T - s), \quad \forall (t, s) \in I \times I,$$

or, which is the same, using Lemma 2.1.2,

$$G_N[T](t, s) = G_P[2T](t, s) + G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.1)$$

The previous expression lets us obtain the exact value at every point of the Green's function of the Neumann problem by means of the values of the periodic one, as long as both Green's functions exist.

Analogously, assuming \tilde{L} is nonresonant on

$$X_{N,2T} = \left\{ u \in W^{2n,1}(J) : u^{(2k+1)}(0) = u^{(2k+1)}(2T) = 0, \, k = 0, \dots, n-1 \right\},$$

it can be also seen that $v \in X_{N,2T}$, that is, v is a solution of problem $(N, 2T)$. Thus, denoting by $G_N[2T]$ the Green's function related to $(N, 2T)$ and arguing as in the previous case, it can be deduced that

$$G_N[T](t, s) = G_N[2T](t, s) + G_N[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \quad (2.2.2)$$

or, using (2.2.1),

$$\begin{aligned}
 G_N[T](t, s) &= G_P[4T](t, s) + G_P[4T](4T - t, s) \\
 &\quad + G_P[4T](2T - t, s) + G_P[4T](2T + t, s),
 \end{aligned} \quad (2.2.3)$$

for all $(t, s) \in I \times I$.

2.2.2. Dirichlet Problem

Now, we will do an analogous decomposition for the Green's function related to problem (D, T) .

To this end, we will assume that operator L is nonresonant on

$$X_{D,T} = \left\{ u \in W^{2n,1}(I) : u^{(2k)}(0) = u^{(2k)}(T) = 0, k = 0, \dots, n-1 \right\},$$

that is, problem (D, T) has a unique solution in $W^{2n,1}(I)$ for all $\sigma \in L^1(I)$. Again, we will also assume that \tilde{L} is nonresonant on $X_{P,2T}$.

Now, if u is the unique solution of (D, T) and we define v as its odd extension to the interval J , it can be seen that $v \in W^{2n,1}(J)$ satisfies the equation $\tilde{L}v(t) = \bar{\sigma}(t)$ for the particular case of taking $\bar{\sigma}$ as the odd extension of σ . Indeed, for $t \in [0, T]$, it is obvious that

$$\tilde{L}v(t) = Lu(t) = \sigma(t) = \bar{\sigma}(t),$$

and, for $t \in [T, 2T]$,

$$\begin{aligned} \tilde{L}v(t) &= v^{(2n)}(t) + \hat{a}_{2n-1}(t)v^{(2n-1)}(t) + \tilde{a}_{2n-2}(t)v^{(2n-2)}(t) \\ &\quad + \dots + \hat{a}_1(t)v'(t) + \tilde{a}_0(t)v(t) \\ &= -u^{(2n)}(2T-t) + \hat{a}_{2n-1}(t)u^{(2n-1)}(2T-t) - \tilde{a}_{2n-2}(t)u^{(2n-2)}(2T-t) \\ &\quad + \dots + \hat{a}_1(t)u'(2T-t) - \tilde{a}_0(t)u(2T-t) \\ &= -u^{(2n)}(2T-t) - \hat{a}_{2n-1}(2T-t)u^{(2n-1)}(2T-t) \\ &\quad - \tilde{a}_{2n-2}(2T-t)u^{(2n-2)}(2T-t) \\ &\quad - \dots - \hat{a}_1(2T-t)u'(2T-t) - \tilde{a}_0(2T-t)u(2T-t) \\ &= -\bar{\sigma}(2T-t) = \bar{\sigma}(t). \end{aligned}$$

Moreover, $v \in X_{P,2T}$ and thus v is a solution of problem $(P, 2T)$.

Therefore, denoting by $G_D[T]$ the Green's function related to (D, T) and reasoning analogously to the previous case, we obtain the following equality:

$$G_D[T](t, s) = G_P[2T](t, s) - G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.4)$$

On the other hand, assuming that \tilde{L} is nonresonant on

$$X_{D,2T} = \left\{ u \in W^{2n,1}(J) : u^{(2k)}(0) = u^{(2k)}(2T) = 0, k = 0, \dots, n-1 \right\},$$

it can also be seen that $v \in X_{D,2T}$, which implies that v is also a solution of $(D, 2T)$. Then, denoting by $G_D[2T]$ the Green's function related to $(D, 2T)$, it can be deduced that

$$G_D[T](t, s) = G_D[2T](t, s) - G_D[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \quad (2.2.5)$$

or, using (2.2.4),

$$\begin{aligned} G_D[T](t, s) = & G_P[4T](t, s) - G_P[4T](4T - t, s) \\ & - G_P[4T](2T - t, s) + G_P[4T](2T + t, s), \end{aligned} \quad (2.2.6)$$

for all $(t, s) \in I \times I$.

2.2.3. Mixed Problems

The same arguments of the two previous subsections are applicable to problems (M_1, T) and (M_2, T) , by assuming the nonresonant character of operator L on

$$X_{M_1, T} = \left\{ u \in W^{2n,1}(I) : u^{(2k+1)}(0) = u^{(2k)}(T) = 0, k = 0, \dots, n-1 \right\}$$

or

$$X_{M_2, T} = \left\{ u \in W^{2n,1}(I) : u^{(2k)}(0) = u^{(2k+1)}(T) = 0, k = 0, \dots, n-1 \right\},$$

respectively. However, these problems will not be related to periodic ones but to the antiperiodic problem $(A, 2T)$. Therefore, we will assume for both cases that operator \tilde{L} is nonresonant on

$$X_{A, 2T} = \left\{ u \in W^{2n,1}(J) : u^{(k)}(0) = -u^{(k)}(2T), k = 0, \dots, 2n-1 \right\}.$$

For (M_1, T) , considering the even extension of its solution and reasoning as in the case of (N, T) , we arrive at the following decomposition:

$$G_{M_1}[T](t, s) = G_A[2T](t, s) - G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.7)$$

As it occurred with $G_N[T]$, $G_{M_1}[T]$ can also be related to $G_N[2T]$:

$$G_{M_1}[T](t, s) = G_N[2T](t, s) - G_N[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \quad (2.2.8)$$

or, using (2.2.1),

$$\begin{aligned} G_{M_1}[T](t, s) = & G_P[4T](t, s) + G_P[4T](4T - t, s) \\ & - G_P[4T](2T - t, s) - G_P[4T](2T + t, s), \end{aligned} \quad (2.2.9)$$

for all $(t, s) \in I \times I$.

Finally, for (M_2, T) , the odd extension of its solution will be considered (as in the case of (D, T)) and the following expression is deduced:

$$G_{M_2}[T](t, s) = G_A[2T](t, s) + G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.10)$$

In this case, $G_{M_2}[T]$ is also related to $G_D[2T]$ in the following way

$$G_{M_2}[T](t, s) = G_D[2T](t, s) + G_D[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \quad (2.2.11)$$

or, using (2.2.4),

$$\begin{aligned} G_{M_2}[T](t, s) &= G_P[4T](t, s) - G_P[4T](4T - t, s) \\ &\quad + G_P[4T](2T - t, s) - G_P[4T](2T + t, s), \end{aligned} \quad (2.2.12)$$

for all $(t, s) \in I \times I$.

On the other hand, it is also possible to obtain a direct relation between the Green's functions of the two mixed problems.

Consider the following operator defined from L by taking the reflection of the coefficients

$$\check{L}u(t) = u^{(2n)}(t) + \sum_{k=0}^{2n-1} (-1)^k a_k(T - t) u^{(k)}(t),$$

for all $t \in I$, and let $\check{G}_{M_2}[T]$ be the Green's function related to the Mixed problem 2 associated with \check{L} , namely,

$$\begin{cases} \check{L}u(t) = \check{\sigma}(t), & t \in I, \\ u^{(2k)}(0) = u^{(2k+1)}(T) = 0, & k = 0, \dots, n-1. \end{cases} \quad (2.2.13)$$

Now, let u be the unique solution of problem (M_1, T) , given explicitly by

$$u(t) = \int_0^T G_{M_1}[T](t, s) \sigma(s) \, ds.$$

If we define $v(t) = u(T - t)$, it is easy to check that v is a solution of problem (2.2.13) for the particular case of taking $\check{\sigma}(t) = \sigma(T - t)$. Therefore,

$$v(t) = \int_0^T \check{G}_{M_2}[T](t, s) \sigma(T - s) \, ds$$

and, making a suitable change of variable,

$$v(t) = \int_0^T \check{G}_{M_2}[T](t, T - s) \sigma(s) \, ds.$$

On the other hand,

$$v(t) = u(T - t) = \int_0^T G_{M_1}[T](T - t, s) \sigma(s) \, ds.$$

Since previous equalities are valid for all $\sigma \in L^1(I)$, we deduce that

$$G_{M_1}[T](T-t, s) = \check{G}_{M_2}[T](t, T-s), \quad \forall (t, s) \in I \times I$$

or, which is the same,

$$G_{M_1}[T](T-t, T-s) = \check{G}_{M_2}[T](t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.14)$$

Analogously, if we denote by $\check{G}_{M_1}[T]$ the Green's function related to the Mixed problem 1 associated with \check{L} , namely,

$$\begin{cases} \check{L}u(t) = \check{\sigma}(t), & t \in I, \\ u^{(2k+1)}(0) = u^{(2k)}(T) = 0, & k = 0, \dots, n-1, \end{cases} \quad (2.2.15)$$

and we repeat the previous reasoning, we reach to the following connecting expression

$$G_{M_2}[T](T-t, T-s) = \check{G}_{M_1}[T](t, s), \quad \forall (t, s) \in I \times I. \quad (2.2.16)$$

2.2.4. Connecting Relations Between Different Problems

On the other hand, assuming again the nonresonant character of all the considered operators on the corresponding spaces, if we sum different combinations of the previous equalities, we obtain more connecting expressions between the considered Green's functions. These expressions will be the basic tool for the spectra decomposition to be developed in Section 2.3. The results are the following:

- From (2.2.1) and (2.2.4), it is deduced that

$$\begin{aligned} G_P[2T](t, s) &= \frac{1}{2} (G_N[T](t, s) + G_D[T](t, s)), \\ G_P[2T](2T-t, s) &= \frac{1}{2} (G_N[T](t, s) - G_D[T](t, s)), \end{aligned} \quad (2.2.17)$$

for all $(t, s) \in I \times I$.

- From (2.2.7) and (2.2.10), we have that

$$\begin{aligned} G_A[2T](t, s) &= \frac{1}{2} (G_{M_2}[T](t, s) + G_{M_1}[T](t, s)), \\ G_A[2T](2T-t, s) &= \frac{1}{2} (G_{M_2}[T](t, s) - G_{M_1}[T](t, s)), \end{aligned} \quad (2.2.18)$$

for all $(t, s) \in I \times I$.

- From (2.2.2) and (2.2.8), it follows that

$$\begin{aligned} G_N[2T](t, s) &= \frac{1}{2} (G_N[T](t, s) + G_{M_1}[T](t, s)), \\ G_N[2T](2T - t, s) &= \frac{1}{2} (G_N[T](t, s) - G_{M_1}[T](t, s)), \end{aligned} \quad (2.2.19)$$

for all $(t, s) \in I \times I$.

- From (2.2.5) and (2.2.11), it is obtained that

$$\begin{aligned} G_D[2T](t, s) &= \frac{1}{2} (G_{M_2}[T](t, s) + G_D[T](t, s)), \\ G_D[2T](2T - t, s) &= \frac{1}{2} (G_{M_2}[T](t, s) - G_D[T](t, s)), \end{aligned} \quad (2.2.20)$$

for all $(t, s) \in I \times I$.

- From (2.2.3), (2.2.6), (2.2.9) and (2.2.12), it is concluded that

$$G_P[4T](t, s) = \frac{1}{4} (G_N[T](t, s) + G_D[T](t, s) + G_{M_1}[T](t, s) + G_{M_2}[T](t, s)),$$

for all $(t, s) \in I \times I$.

2.3. Decomposition of the Spectra

In this section we will show how the spectra of the considered problems can be connected.

We will denote by $\Lambda_N[T]$, $\Lambda_D[T]$, $\Lambda_{M_1}[T]$, $\Lambda_{M_2}[T]$, $\Lambda_P[2T]$, $\Lambda_A[2T]$, $\Lambda_N[2T]$, $\Lambda_D[2T]$ and $\Lambda_P[4T]$ the set of eigenvalues of problems (N, T) , (D, T) , (M_1, T) , (M_2, T) , $(P, 2T)$, $(A, 2T)$, $(N, 2T)$, $(D, 2T)$ and $(P, 4T)$, respectively.

From equality (2.2.1) we have that if problem $(P, 2T)$ has a unique solution, then problem (N, T) has a solution given by

$$u(t) = \int_0^T (G_P[2T](t, s) + G_P[2T](2T - t, s)) \sigma(s) \, ds.$$

The uniqueness of this solution follows from the fact that the Neumann boundary conditions are linearly independent (see [18, Lemma 1.2.21]).

Consequently we observe that if operator \tilde{L} is nonresonant on $X_{P,2T}$, then operator L is nonresonant on $X_{N,T}$. In other words, the sequence of eigenvalues of problem (N, T) is contained into the sequence of eigenvalues of $(P, 2T)$.

The same argument is valid, by means of equality (2.2.4), to ensure that if problem $(P, 2T)$ has a unique solution, then problem (D, T) has a unique solution too.

On the other hand, (2.2.17) implies that the uniqueness of solution of both problems (N, T) and (D, T) warrants the uniqueness of solution of $(P, 2T)$.

Thus, we conclude that the spectrum of problem $(P, 2T)$ is the union of the spectra of (N, T) and (D, T) , that is,

$$\Lambda_N[T] \cup \Lambda_D[T] = \Lambda_P[2T].$$

Using analogous arguments, from (2.2.2), (2.2.8) and (2.2.19), we deduce that

$$\Lambda_N[T] \cup \Lambda_{M_1}[T] = \Lambda_N[2T];$$

from (2.2.5), (2.2.11) and (2.2.20),

$$\Lambda_D[T] \cup \Lambda_{M_2}[T] = \Lambda_D[2T];$$

from (2.2.7), (2.2.10) and (2.2.18),

$$\Lambda_{M_1}[T] \cup \Lambda_{M_2}[T] = \Lambda_A[2T];$$

and from (2.2.3), (2.2.6), (2.2.9) and (2.2.12),

$$\Lambda_N[T] \cup \Lambda_D[T] \cup \Lambda_{M_1}[T] \cup \Lambda_{M_2}[T] = \Lambda_P[4T].$$

Finally, if we denote by $\check{\Lambda}_{M_2}[T]$ and $\check{\Lambda}_{M_1}[T]$ the set of eigenvalues of problems (2.2.13) and (2.2.15), respectively, from (2.2.14) and (2.2.16) we deduce that

$$\Lambda_{M_1}[T] = \check{\Lambda}_{M_2}[T]$$

and

$$\Lambda_{M_2}[T] = \check{\Lambda}_{M_1}[T].$$

As an immediate consequence we have the following result.

Corollary 2.3.1. *If $a_k(t) = (-1)^k a_k(T - t)$ for all $k = 0, \dots, 2n - 1$, then the spectra of the two mixed problems coincide, that is,*

$$\Lambda_{M_1}[T] = \Lambda_{M_2}[T].$$

Moreover, if we denote by $\lambda_0^N[T]$, $\lambda_0^D[T]$, $\lambda_0^{M_1}[T]$, $\lambda_0^{M_2}[T]$, $\lambda_0^P[2T]$, $\lambda_0^A[2T]$, $\lambda_0^N[2T]$, $\lambda_0^D[2T]$ and $\lambda_0^P[4T]$ the first eigenvalue of problems (N, T) , (D, T) , (M_1, T) , (M_2, T) , $(P, 2T)$, $(A, 2T)$, $(N, 2T)$, $(D, 2T)$ and $(P, 4T)$, respectively, from the connecting expressions proved in Section 2.2, we will deduce the relations below.

Theorem 2.3.2. *Assume that all the previously considered spectra are not empty, the first eigenvalue of each problem (except for $(A, 2T)$) is simple and its related eigenfunction has constant sign. Then, the following equalities are fulfilled for any $a_0, \dots, a_{2n-1} \in L^1(I)$:*

1. $\lambda_0^N[T] = \lambda_0^P[2T] < \lambda_0^D[T]$.
2. $\lambda_0^N[T] = \lambda_0^N[2T] < \lambda_0^{M_1}[T]$.
3. $\lambda_0^N[T] = \lambda_0^P[4T]$.
4. $\lambda_0^{M_2}[T] = \lambda_0^D[2T] < \lambda_0^D[T]$.
5. $\lambda_0^N[T] < \lambda_0^{M_2}[T]$.
6. $\lambda_0^A[2T] = \min \{ \lambda_0^{M_1}[T], \lambda_0^{M_2}[T] \}$.

Proof. Assertion 1 is proved in the following way: as we have seen above, the spectrum of $(P, 2T)$ is decomposed as $\Lambda_P[2T] = \Lambda_N[T] \cup \Lambda_D[T]$, which implies that

$$\lambda_0^P[2T] = \min \{ \lambda_0^N[T], \lambda_0^D[T] \}.$$

Consider now the even extension to J of the eigenfunction associated to $\lambda_0^N[T]$. This extension has constant sign on J and, moreover, it satisfies periodic boundary conditions, so it is a constant sign eigenfunction of $(P, 2T)$. On the contrary, the odd extension to J of the eigenfunction associated to $\lambda_0^D[T]$ is a sign changing eigenfunction of $(P, 2T)$. Therefore, since we have assumed that the eigenfunction related to the first eigenvalue of each problem has constant sign, we deduce that $\lambda_0^N[T] = \lambda_0^P[2T] < \lambda_0^D[T]$.

An analogous argument is valid to prove Assertion 2, by taking into account that $\Lambda_N[2T] = \Lambda_N[T] \cup \Lambda_{M_1}[T]$.

Assertion 3 is deduced from the two previous one. Indeed, Assertion 1 implies that $\lambda_0^N[2T] = \lambda_0^P[4T]$ and, from Assertion 2, we deduce the equality.

Assertion 4 is proved analogously to Assertions 1 and 2, taking into account the decomposition $\Lambda_D[2T] = \Lambda_D[T] \cup \Lambda_{M_2}[T]$.

Now Assertion 5 can be deduced from 1, 2 and 4. Indeed, Assertion 1 implies that $\lambda_0^N[2T] < \lambda_0^D[2T]$ and, using Assertions 2 and 4,

$$\lambda_0^N[T] = \lambda_0^N[2T] < \lambda_0^D[2T] = \lambda_0^{M_2}[T].$$

Finally, Assertion 6 is an immediate consequence of

$$\Lambda_A[2T] = \Lambda_{M_1}[T] \cup \Lambda_{M_2}[T].$$

□

Remark 2.3.3. *With respect to the hypothesis that all the considered spectra are not empty note that, as a consequence of the relations proved at the beginning of this section, if one of those spectra is not empty, we could ensure that some others are not empty too.*

On the other hand, there are several results which ensure that, under some suitable conditions, the first eigenvalue of a boundary value problem is simple and its related eigenfunction has constant sign, for instance, Krein-Rutman Theorem.

Sufficient conditions to ensure that all the hypotheses required in previous theorem are fulfilled can be found in [87].

First, we can deduce from Theorem 1 in such reference that if there exists some λ for which the Green's function $G[\lambda, T]$ has constant sign and the spectrum of such problem is not empty, then the eigenfunction related to the first eigenvalue has constant sign.

Moreover, from Theorem 2 in [87] it is deduced that if there exists some λ for which the Green's function $G[\lambda, T]$ has strict constant sign on $[0, T] \times (0, T)$ then the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has strict constant sign on $(0, T)$.

Finally, from Theorem 2' in [87] we can ensure that if there exists some λ for which $G[\lambda, T]$ has strict constant sign on $(0, T) \times (0, T)$ and there exists a continuous function ϕ , positive on $(0, T)$, such that

$$\frac{G[\lambda, T](t, s)}{\phi(t)}$$

is continuous on $[0, T] \times [0, T]$ and positive on $[0, T] \times (0, T)$, then the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has strict constant sign on $(0, T)$.

Analogously, if conditions given in Lemmas 1.1.8 or 1.1.9 hold for some λ , then we are also able to deduce that the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has constant sign. Details of this can be seen in [18], where it is proved that Lemmas 1.1.8 or 1.1.9 imply that Krein-Rutman Theorem holds.

Finally, we must note that, since the eigenfunctions of the considered problems are related, the constant sign of the eigenfunction associated with the first eigenvalue of a problem implies (in some cases) the constant sign of the eigenfunction of other problems.

2.4. Constant Sign of Green's Functions

From all the connecting expressions between different Green's functions given in Section 2.2, it is possible to deduce that the constant sign of one of them implies the

constant sign of another one.

In particular, from (2.2.1), (2.2.2) and (2.2.11) we deduce the relations below.

Corollary 2.4.1. *The following properties hold for any $a_0, \dots, a_{2n-1} \in L^1(I)$:*

1. *If $G_P[2T] \leq 0$ on $J \times J$, then $G_N[T] \leq 0$ on $I \times I$.*
2. *If $G_P[2T] \geq 0$ on $J \times J$, then $G_N[T] \geq 0$ on $I \times I$.*
3. *If $G_N[2T] \leq 0$ on $J \times J$, then $G_N[T] \leq 0$ on $I \times I$.*
4. *If $G_N[2T] \geq 0$ on $J \times J$, then $G_N[T] \geq 0$ on $I \times I$.*
5. *If $G_D[2T] \leq 0$ on $J \times J$, then $G_{M_2}[T] \leq 0$ on $I \times I$.*
6. *If $G_D[2T] \geq 0$ on $J \times J$, then $G_{M_2}[T] \geq 0$ on $I \times I$.*

Remark 2.4.2. *In the particular case of considering disconjugated operators, the values of λ for which some of the previously considered Green's functions, related to operator $L[\lambda]$, have constant sign have been characterized in [38, 39]. More specifically, the general boundary conditions considered in that reference include what we have called Dirichlet and Mixed conditions, but do not cover neither Neumann nor periodic and antiperiodic conditions.*

The reciprocal of Assertions 1 and 2 in the previous corollary holds for constant coefficients. This occurs as a consequence of the following property.

Lemma 2.4.3. *[18, Section 1.4] Let*

$$L_n u(t) \equiv u^{(n)}(t) + a_{n-1}(t) u_{n-1}(t) + \dots + a_1(t) u'(t) + a_0(t) u(t), \quad t \in I,$$

be a n -th order linear operator and let $G_P[T]$ denote the Green's function related to the periodic problem

$$\begin{cases} L_n u(t) = 0, & t \in I, \\ u^{(k)}(0) = u^{(k)}(T), & k = 0, \dots, n-1. \end{cases}$$

If the coefficients a_k , $k = 0, \dots, n-1$, involved in the definition of operator L_n are constant on I , then the Green's function is constant over the straight lines of slope one, that is, it satisfies the following property

$$G_P[T](t, s) = \begin{cases} G_P[T](t-s, 0), & 0 \leq s \leq t \leq T, \\ G_P[T](T+t-s, 0), & \text{otherwise.} \end{cases}$$

As a consequence, we arrive at the following result.

Theorem 2.4.4. *If all the coefficients a_0, \dots, a_{2n-1} are constant, then the following properties hold:*

1. $G_P[2T] \leq 0$ on $J \times J$ if and only if $G_N[T] \leq 0$ on $I \times I$.
2. $G_P[2T] \geq 0$ on $J \times J$ if and only if $G_N[T] \geq 0$ on $I \times I$.

Proof. From Corollary 2.4.1, the assertion is equivalent to prove that if $G_P[2T]$ changes sign, then $G_N[T]$ will also change sign. Indeed, assume that there exist two pairs of values (t_1, s_1) and (t_2, s_2) such that

$$G_P[2T](t_1, s_1) < 0 \quad \text{and} \quad G_P[2T](t_2, s_2) > 0.$$

As it is satisfied that $G_P[2T](t, s) = G_P[2T](s, t)$ for all $(t, s) \in J \times J$, we may assume, without loss of generality, that $s_1 \leq t_1$ and $s_2 \leq t_2$.

Since all the coefficients a_0, \dots, a_{2n-1} are constant then, from Lemma 2.4.3, it holds that

$$G_P[2T](t, s) = \begin{cases} G_P[2T](t - s, 0), & 0 \leq s \leq t \leq 2T, \\ G_P[2T](2T + t - s, 0), & \text{otherwise.} \end{cases}$$

Therefore, it is fulfilled that

$$G_P[2T](t_1, s_1) = G_P[2T](t_1 - s_1, 0)$$

and

$$G_P[2T](t_2, s_2) = G_P[2T](t_2 - s_2, 0).$$

On the other hand, from equality (2.1.2) and the fact that the Green's function satisfies the periodic boundary conditions (see Definition 1.1.1), it holds that

$$G_P[2T](t_1 - s_1, 0) = G_P[2T](2T - t_1 + s_1, 2T) = G_P[2T](2T - t_1 + s_1, 0)$$

and

$$G_P[2T](t_2 - s_2, 0) = G_P[2T](2T - t_2 + s_2, 2T) = G_P[2T](2T - t_2 + s_2, 0).$$

Now, we will distinguish two possibilities:

- If $t_1 - s_1 \leq T$, then

$$\begin{aligned} G_N[T](t_1 - s_1, 0) &= G_P[2T](t_1 - s_1, 0) + G_P[2T](2T - t_1 + s_1, 0) \\ &= 2 G_P[2T](t_1 - s_1, 0) < 0. \end{aligned}$$

- When $t_1 - s_1 > T$, we have

$$\begin{aligned} G_N[T](2T - t_1 + s_1, 0) &= G_P[2T](2T - t_1 + s_1, 0) + G_P[2T](t_1 - s_1, 0) \\ &= 2G_P[2T](t_1 - s_1, 0) < 0. \end{aligned}$$

Analogously, if $t_2 - s_2 \leq T$, then

$$G_N[T](t_2 - s_2, 0) = 2G_P[2T](t_2 - s_2, 0) > 0$$

and, if $t_2 - s_2 > T$, then

$$G_N[T](2T - t_2 + s_2, 0) = 2G_P[2T](t_2 - s_2, 0) > 0.$$

It is clear that, in any of the cases, $G_N[T]$ changes its sign and the result holds. \square

The following counterexample shows that the converse of Assertion 2 in Corollary 2.4.1 is not true in general for nonconstant coefficients.

Example 2.4.5. Consider the Neumann problem on $[0, T] = [0, 2]$ related to operator

$$L u(t) = u^{(4)}(t) + ((t - 2)^4 + \lambda) u(t), \quad t \in [0, 2], \quad (2.4.1)$$

and the periodic problem on $[0, 2T] = [0, 4]$ related to

$$\tilde{L} u(t) \equiv u^{(4)}(t) + ((t - 2)^4 + \lambda) u(t), \quad t \in [0, 4]. \quad (2.4.2)$$

By numerical approach, we find that $G_N[T]$ is nonpositive for $\lambda \in (\lambda_1, \lambda_0^N[T])$, where $\lambda_1 \approx -2.26$ and $\lambda_0^N[T] = \lambda_0^P[2T] \approx -1.746$. Moreover, it is nonnegative for $\lambda \in (\lambda_0^N[T], \lambda_2)$, with $\lambda_2 \approx 4.11$.

However, $G_P[2T]$ is nonpositive for $\lambda \in (\lambda_1, \lambda_0^P[2T])$ and nonnegative for $\lambda \in (\lambda_0^P[2T], \lambda_3)$, with $\lambda_3 \approx 5.95$.

Despite this, we remark that the interval of values of λ for which $G_N[T]$ and $G_P[2T]$ are nonpositive is exactly the same.

Remark 2.4.6. It must be pointed out that the converse of Assertion 2 in Corollary 2.4.1 also holds for several examples with non constant coefficients. However we have not been able to prove the existence of any general condition under which this assertion holds.

Furthermore, up to this moment, we have not been able to find a counterexample for the converse of Assertion 1. So, it remains as an open problem to know if Assertion 1 is or not an equivalence for $n \geq 2$.

The following counterexample shows that the converse of Assertions 3 and 4 in Corollary 2.4.1 does not hold in general, not even in the constant case.

Example 2.4.7. Consider the following Neumann problem with constant coefficients on $[0, T] = [0, \frac{3}{2}]$ related to the following operator

$$L u(t) \equiv u^{(4)}(t) + \lambda u(t), \quad t \in \left[0, \frac{3}{2}\right],$$

and the Neumann problem on $[0, 2T] = [0, 3]$ related to

$$\tilde{L} u(t) \equiv u^{(4)}(t) + \lambda u(t), \quad t \in [0, 3],$$

By numerical approach, it can be seen that in this case $G_N[T]$ is nonpositive for $\lambda \in (\lambda_4, \lambda_0^N[T])$, with $\lambda_4 \approx -6.1798$ and $\lambda_0^N[T] = 0$, and nonnegative for $\lambda \in (\lambda_0^N[T], \lambda_5)$, with $\lambda_5 \approx 24.7192$.

However, $G_N[2T]$ is nonpositive for $\lambda \in (\lambda_6, \lambda_0^N[2T])$, with $\lambda_6 \approx -0.3862$ and $\lambda_0^N[2T] = 0$, and nonnegative for $\lambda \in (\lambda_0^N[2T], \lambda_7)$, with $\lambda_7 \approx 1.5449$.

So, the converse of Assertions 3 and 4 does not hold for these operators.

The following counterexample shows that the converse of Assertions 5 and 6 in Corollary 2.4.1 is not true in general, not even in the constant case.

Example 2.4.8. Consider the Mixed problem 2 with constant coefficients on the interval $[0, T] = [0, 1]$ related to operator

$$L u(t) \equiv u^{(4)}(t) + \lambda u(t), \quad t \in [0, 1],$$

and the Dirichlet problem on $[0, 2T] = [0, 2]$ related to

$$\tilde{L} u(t) \equiv u^{(4)}(t) + \lambda u(t), \quad t \in [0, 2].$$

In this case, it can be seen that $G_{M_2}[T]$ is nonpositive for $\lambda \in (\lambda_8, \lambda_0^{M_2}[T])$, with $\lambda_8 \approx -31.2852$ and $\lambda_0^{M_2}[T] = \lambda_0^D[2T] = -\frac{\pi^4}{16} \approx -6.088$. Moreover, it is nonnegative for $\lambda \in (\lambda_0^{M_2}[T], \lambda_9)$, with $\lambda_9 \approx 389.6365$.

However, $G_D[2T]$ is nonpositive for $\lambda \in (\lambda_{10}, \lambda_0^D[2T])$, with $\lambda_{10} \approx -14.8576$, and nonnegative for $\lambda \in (\lambda_0^D[2T], \lambda_{11})$, with $\lambda_{11} \approx 59.4303$.

Finally, from the relations given in Theorem 2.3.2, together with the general characterization given in Lemmas 1.1.8 and 1.1.9, we can deduce the following corollary.

To establish the suitable conditions under which next result is valid, we need to introduce some notation. This way, analogously to what we have done in Section 1.1,

consider the parametrized operators defined from L or \tilde{L} . In particular, we will denote by $L[\lambda] u(t) \equiv L u(t) + \lambda u(t)$. In this case, to stress also its dependence on λ , we will denote by $G[\lambda, T]$ the Green's function related to $L[\lambda]$, which will also have the corresponding subscript when we refer to one particular problem. Analogous notation can be used for $\tilde{L}[\lambda]$ and $\tilde{\tilde{L}}[\lambda]$, whose related Green's functions will be denoted by $G[\lambda, 2T]$ and $G[\lambda, 4T]$, respectively.

Corollary 2.4.9. *Assume that we are in conditions to apply Lemmas 1.1.8 and 1.1.9, that is, all the considered Green's functions $G[\lambda, T]$ (or $G[\lambda, 2T]$, $G[\lambda, 4T]$, with the suitable subscript for each case) are:*

- *nonpositive on $I \times I$ if and only if $\lambda \in (-\infty, \lambda_1)$ or $\lambda \in [-\bar{\mu}, \lambda_1)$, with $\lambda_1 > 0$ the first eigenvalue of operator L_n coupled with the corresponding boundary conditions and $\bar{\mu} \geq 0$ such that $L_n[-\bar{\mu}]$ is nonresonant on X and the related nonpositive Green's function $G[-\bar{\mu}]$ vanishes at some point of the square $I \times I$.*
- *nonnegative on $I \times I$ if and only if $\lambda \in (\lambda_1, \infty)$ or $\lambda \in (\lambda_1, \bar{\mu}]$, with $\lambda_1 < 0$ the first eigenvalue of operator L_n coupled with the corresponding boundary conditions and $\bar{\mu} \geq 0$ such that $L_n[\bar{\mu}]$ is nonresonant on X and the related nonnegative Green's function $G[\bar{\mu}]$ vanishes at some point of the square $I \times I$.*

Then the following relations between the constant sign of Green's functions are valid for any $a_0, \dots, a_{2n-1} \in L^1(I)$:

- *If $G_N[T]$ is nonpositive on $I \times I$, then $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_N[2T]$ is nonpositive on $J \times J$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_P[2T]$ is nonpositive on $J \times J$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_P[4T]$ is nonpositive on $[0, 4T] \times [0, 4T]$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_{M_2}[T]$ is nonpositive on $I \times I$, then $G_D[T]$ either changes sign or is nonpositive on $I \times I$.*
- *If $G_D[2T]$ is nonpositive on $J \times J$, then $G_D[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*

2.5. Comparison Principles

In this section we will use the connecting expressions for Green's functions obtained in Section 2.2 to compare the values that several Green's functions take point by point.

First, from (2.2.17), under the hypothesis of the constant sign of $G_P[2T]$, we obtain the following comparison between Green's functions of problems (N, T) and (D, T) .

Corollary 2.5.1. *If $G_P[2T] \geq 0$ on $J \times J$, then*

$$G_N[T](t, s) \geq |G_D[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

If $G_P[2T] \leq 0$ on $J \times J$, then

$$G_N[T](t, s) \leq -|G_D[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

As a consequence, we can compare the solutions of (N, T) and (D, T) , as follows.

Theorem 2.5.2. *Let u_N be the unique solution of problem (N, T) for $\sigma = \sigma_1$ and u_D the unique solution of problem (D, T) for $\sigma = \sigma_2$. Then*

1. *If $G_P[2T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a. e. $t \in I$, then $|u_D(t)| \leq u_N(t)$ for all $t \in I$.*
2. *If $G_P[2T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then $u_N(t) \leq 0$ and $u_N(t) \leq u_D(t)$ for all $t \in I$.*
3. *If $G_P[2T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then $u_N(t) \geq 0$ and $u_D(t) \leq u_N(t)$ for all $t \in I$.*

Proof. 1. Since $G_P[2T] \geq 0$ on $J \times J$ then, from Corollary 2.5.1, it holds that

$$\begin{aligned} |u_D(t)| &= \left| \int_0^T G_D[T](t, s) \sigma_2(s) \, ds \right| \leq \int_0^T |G_D[T](t, s)| |\sigma_2(s)| \, ds \\ &\leq \int_0^T G_N[T](t, s) \sigma_1(s) \, ds = u_N(t). \end{aligned}$$

2. Since $G_P[2T] \leq 0$ on $J \times J$ then, from Corollary 2.5.1, since $\sigma_1(s) \geq 0$ a. e. $s \in I$, we have that

$$G_N[T](t, s) \sigma_1(s) \leq -|G_D[T](t, s)| \sigma_1(s), \quad \forall (t, s) \in I \times I.$$

Moreover, from $\sigma_2(s) \leq \sigma_1(s)$ a. e. $s \in I$, we deduce that

$$-|G_D[T](t, s)| \sigma_1(s) \leq -|G_D[T](t, s)| \sigma_2(s), \quad \forall (t, s) \in I \times I.$$

Finally, since $\sigma_2(s) \geq 0$ a. e. $s \in I$,

$$-|G_D[T](t, s)| \sigma_2(s) \leq G_D[T](t, s) \sigma_2(s), \quad \forall (t, s) \in I \times I.$$

Therefore, for all $t \in I$, we have

$$\begin{aligned} u_N(t) &= \int_0^T G_N[T](t, s) \sigma_1(s) \, ds \leq \int_0^T -|G_D[T](t, s)| \sigma_1(s) \, ds \\ &\leq \int_0^T -|G_D[T](t, s)| \sigma_2(s) \, ds \leq \int_0^T G_D[T](t, s) \sigma_2(s) \, ds = u_D(t). \end{aligned}$$

Finally, the fact that $u_N \leq 0$ on I is a direct consequence from $G_N[T] \leq 0$ and $\sigma_1 \geq 0$.

3. Since $G_P[2T] \leq 0$ on $J \times J$ then, from Corollary 2.5.1, it can be deduced that

$$G_N[T](t, s) \leq G_D[T](t, s) \text{ and } G_N[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I$$

and so, since $\sigma_2(s) \leq 0$ a. e. $s \in I$,

$$G_D[T](t, s) \sigma_2(s) \leq G_N[T](t, s) \sigma_2(s), \quad \forall (t, s) \in I \times I$$

and, from $\sigma_1(s) \leq \sigma_2(s)$ a. e. $s \in I$, we deduce that

$$G_N[T](t, s) \sigma_2(s) \leq G_N[T](t, s) \sigma_1(s), \quad \forall (t, s) \in I \times I.$$

Therefore,

$$\begin{aligned} u_D(t) &= \int_0^T G_D[T](t, s) \sigma_2(s) \, ds \leq \int_0^T G_N[T](t, s) \sigma_2(s) \, ds \\ &\leq \int_0^T G_N[T](t, s) \sigma_1(s) \, ds = u_N(t). \end{aligned}$$

Finally, the fact that $u_N \geq 0$ on I is a direct consequence from $G_N[T] \leq 0$ and $\sigma_1 \leq 0$.

□

The situation described in previous theorem is represented in Figures 2.5.1, 2.5.2 and 2.5.3.

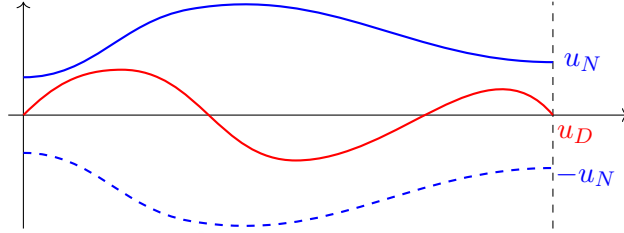


Figure 2.5.1: Solutions of (N, T) and (D, T) in Case 1 in Theorem 2.5.2.

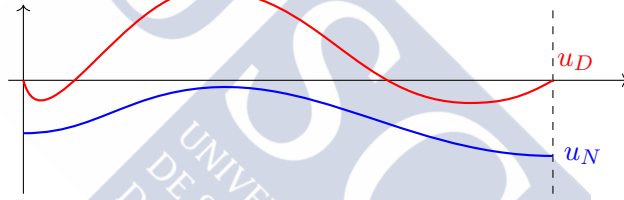


Figure 2.5.2: Solutions of (N, T) and (D, T) in Case 2 in Theorem 2.5.2.

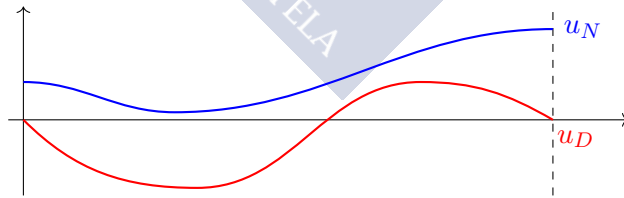


Figure 2.5.3: Solutions of (N, T) and (D, T) in Case 3 in Theorem 2.5.2.

Analogously, from (2.2.19) and (2.2.20), the constant sign of either $G_N[2T]$ or $G_D[2T]$ lets us deduce some point-by-point relation between various Green's functions.

Corollary 2.5.3. 1. If $G_N[2T] \geq 0$ on $J \times J$, then

$$G_N[T](t, s) \geq |G_{M_1}[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

2. If $G_N[2T] \leq 0$ on $J \times J$, then

$$G_N[T](t, s) < -|G_{M_1}[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

3. If $G_D[2T] \leq 0$ on $J \times J$, then

$$G_{M_2}[T](t, s) < -|G_D[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

4. If $G_D[2T] \geq 0$ on $J \times J$, then

$$G_{M_2}[T](t, s) \geq |G_D[T](t, s)|, \quad \forall (t, s) \in I \times I.$$

As a consequence of the previous corollary, we deduce the following comparison principles between the solutions of the corresponding problems. The arguments are similar to the ones used in the proof of Theorem 2.5.2.

Theorem 2.5.4. *Let u_N be the unique solution of problem (N, T) for $\sigma = \sigma_1$ and u_{M_1} the unique solution of problem (M_1, T) for $\sigma = \sigma_2$. Then*

1. *If $G_N[2T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a. e. $t \in I$, then $|u_{M_1}(t)| \leq u_N(t)$ for all $t \in I$.*
2. *If $G_N[2T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then $u_N(t) \leq 0$ and $u_N(t) \leq u_{M_1}(t)$ for all $t \in I$.*
3. *If $G_N[2T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then $u_N(t) \geq 0$ and $u_{M_1}(t) \leq u_N(t)$ for all $t \in I$.*

Theorem 2.5.5. *Let u_{M_2} be the unique solution of problem (M_2, T) for $\sigma = \sigma_1$ and u_D the unique solution of problem (D, T) for $\sigma = \sigma_2$. Then, it holds that*

1. *If $G_D[2T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a. e. $t \in I$, then $|u_D(t)| \leq u_{M_2}(t)$ for all $t \in I$.*
2. *If $G_D[2T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then $u_{M_2}(t) \leq 0$ and $u_{M_2}(t) \leq u_D(t)$ for all $t \in I$.*
3. *If $G_D[2T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then $u_{M_2}(t) \geq 0$ and $u_D(t) \leq u_{M_2}(t)$ for all $t \in I$.*

Chapter 3

Second Order Equation

In this chapter we will particularize the study developed in Chapter 2 for the linear equation of order two, that is, we will consider $n = 1$ in previous chapter. The reason to study this particular case is that, when working with differential equations of order two, Sturm-Liouville's Theory can be applied and this makes it possible to obtain stronger results than in the general case for any arbitrary n .

Obviously, all the results obtained in Chapter 2 are valid in this case. Moreover, we will improve many of them, obtaining for instance stronger relations between the constant sign of different Green's functions and stronger comparison principles. Finally, we will also include some results whose proofs will be based on Sturm-Liouville theory and oscillation properties, and so they can not be extended to the general case considered in Chapter 2. In many of those cases, we will give counterexamples to show that the results in this chapter are not true for higher order equations.

This chapter is divided in three sections. First, Section 3.1 compiles some classical results of Sturm-Liouville theory which will be used later.

In Section 3.2, we will particularize all the results of Chapter 2 in case that $n = 1$ and the coefficient a_1 is identically zero, that is, we will work with operator

$$L u(t) \equiv u''(t) + a(t) u(t), \quad (3.0.1)$$

which is known as Hill's operator. The reason why we have decided to assume that $a_1 \equiv 0$ is the fact that every differential equation of order 2 written in the general form

$$u''(t) + a_1(t) u'(t) + a_0(t) u(t) = 0, \quad (3.0.2)$$

such that the coefficients a_1 and a_0 have enough regularity, can be transformed into one of the type of (3.0.1) by means of a suitable change of variable. The results in this section can be found in [22].

Finally, Section 3.3 considers a more general equation of order 2, namely

$$(p u')'(t) + \bar{a}(t) u(t) = 0,$$

with $p(t) > 0$ a.e. $t \in I$ and $\frac{1}{p} \in L^1(I)$. Using a suitable change of variable, we will prove that this problem is equivalent to Hill's equation and, as a consequence, we will be able to rewrite the results obtained for Hill's equation in terms of this new problem. The results in this section are compiled in [23, Section 3.5].

3.1. Preliminaries: Oscillation and Spectral Theory

Consider the homogeneous linear differential equation of second order

$$y''(t) + a_1(t) y'(t) + a_0(t) y(t) = 0, \quad \text{a. e. } t \in \mathbb{R} \quad (3.1.1)$$

with $a_1, a_0 \in L^\infty(\mathbb{R})$.

This equation is not necessarily solvable in terms of elementary functions. However, it is possible to establish some qualitative properties of its solutions.

First, the next theorem ensures that the zeros of two linearly independent solutions of (3.1.1) must alternate.

Theorem 3.1.1 (Sturm's separation, [133, Section 24, Theorem A]). *Let y_1 and y_2 be two linearly independent solutions of (3.1.1). Neither y_1 and y_2 nor y_1' and y_2' can have any zero in common. Moreover, y_1 vanishes exactly once between two consecutive zeros of y_2 , and reciprocally.*

Remark 3.1.2. *The previous theorem, as it is formulated in [133, Section 24] does not include the fact y_1' and y_2' do not have any zero in common. However, this is an immediate consequence of the proof which can be seen in [23, Theorem 6].*

We will simplify now expression (3.1.1), seeing that every equation in this form, in which a_1 and a_0 satisfy suitable regularity conditions, could be rewritten as a Hill's equation, also called the normal form of (3.1.1),

$$u''(t) + a(t) u(t) = 0, \quad \text{a. e. } t \in \mathbb{R}. \quad (3.1.2)$$

In order to write (3.1.1) in the normal form, we decompose $y(t) = u(t) v(t)$, so that $y' = u v' + u' v$ and $y'' = u v'' + 2 u' v' + u'' v$. Substituting in (3.1.1), we obtain

$$v u'' + (2 v' + a_1 v) u' + (v'' + a_1 v' + a_0 v) u = 0.$$

Making the coefficient of u' equal to zero, we deduce that, for some $t_0 \in \mathbb{R}$,

$$v(t) = e^{-\frac{1}{2} \int_{t_0}^t a_1(s) ds}$$

reduces (3.1.1) into the normal form (3.1.2), with

$$a(t) = \frac{v''(t)}{v(t)} + a_1(t) \frac{v'(t)}{v(t)} + a_0(t).$$

Taking into account that

$$\frac{v'(t)}{v(t)} = -\frac{1}{2} a_1(t)$$

and

$$\frac{v''(t)}{v(t)} = -\frac{1}{2} a_1'(t) + \frac{1}{4} a_1^2(t),$$

we obtain

$$a(t) = a_0(t) - \frac{1}{4} a_1^2(t) - \frac{1}{2} a_1'(t).$$

We observe that, since v does not take the value zero, the transformation we have just made does not affect neither to the zeros of the solutions nor to their oscillation and sign.

Therefore, from now on we will focus our study on the Hill's equation

$$u''(t) + (a(t) + \lambda) u(t) = 0, \quad t \in I. \quad (3.1.3)$$

The following theorem describes the influence that the potential a has on the speed of oscillation of the solutions.

Theorem 3.1.3 (Sturm's comparison, [133, Section 25, Theorem B]). *Let u and v be nontrivial solutions of*

$$u''(t) + q(t) u(t) = 0, \quad a. e. \ t \in \mathbb{R}$$

and

$$v''(t) + r(t) v(t) = 0, \quad a. e. \ t \in \mathbb{R},$$

respectively, with $q, r \in L^1_{loc}(\mathbb{R})$ such that $q > r$ a. e. on \mathbb{R} . Then u vanishes at least once between two consecutive zeros of v .

On the other hand, the Oscillation Theorem establishes a certain relation of order between the eigenvalues of the equation (3.1.3) coupled with periodic and antiperiodic conditions. Before formulating it, we need to introduce two preliminary definitions.

Definition 3.1.4. *Hill's equation (3.1.3) has two solutions, u_1 and u_2 , which are uniquely determined by the following initial conditions:*

$$u_1(0) = 1, \ u_1'(0) = 0,$$

$$u_2(0) = 0, \ u_2'(0) = 1.$$

These solutions are known as normalized solutions. In order to emphasize its dependence on the parameter λ , sometimes we will denote them by $u_1(t, \lambda)$ and $u_2(t, \lambda)$.

Definition 3.1.5. *The function*

$$\Delta(\lambda) = u_1(T, \lambda) + u_2'(T, \lambda)$$

is known as the discriminant of the Hill's equation (3.1.3).

Now we are in conditions to enunciate Oscillation Theorem.

Theorem 3.1.6 (Oscillation). *[106, Chapter 2] There exist two increasing sequences of real numbers*

$$\{\lambda_n^P[T]\}_{n=0}^{\infty} \quad \text{and} \quad \{\lambda_n^A[T]\}_{n=0}^{\infty},$$

such that the equation (3.1.3) has a nontrivial T -periodic solution if and only if $\lambda = \lambda_n^P[T]$, $n = 0, 1, 2, \dots$, and a nontrivial T -antiperiodic solution if and only if $\lambda = \lambda_n^A[T]$, $n = 0, 1, 2, \dots$

Moreover, $\lambda_n^P[T]$, $n = 0, 1, \dots$, are the roots of the equation $\Delta(\lambda) = 2$ and $\lambda_n^A[T]$, $n = 0, 1, \dots$, those of $\Delta(\lambda) = -2$.

The following inequalities hold

$$\lambda_0^P[T] < \lambda_0^A[T] \leq \lambda_1^A[T] < \lambda_1^P[T] \leq \lambda_2^P[T] < \lambda_2^A[T] \leq \lambda_3^A[T] < \lambda_3^P[T] \dots$$

and, moreover,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^P[T]} = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^A[T]} = 0.$$

The trivial solution of (3.1.3) is stable if λ belongs to one of the following intervals

$$(\lambda_0^P[T], \lambda_0^A[T]), (\lambda_1^A[T], \lambda_1^P[T]), (\lambda_2^P[T], \lambda_2^A[T]), (\lambda_3^A[T], \lambda_3^P[T]), \dots$$

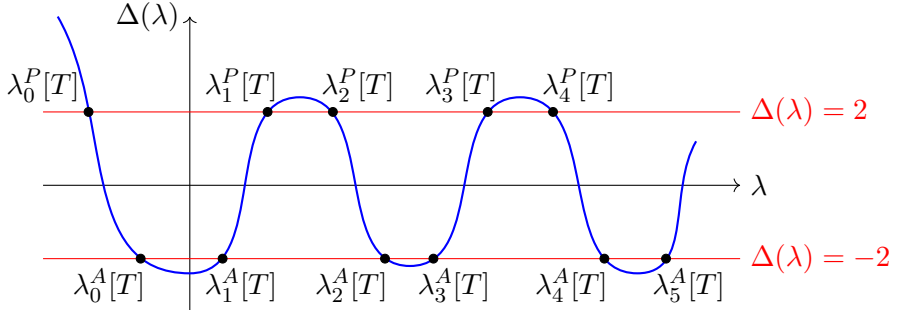
On the other hand, if λ belongs to one of the intervals

$$(-\infty, \lambda_0^P[T]), (\lambda_0^A[T], \lambda_1^A[T]), (\lambda_1^P[T], \lambda_2^P[T]), (\lambda_2^A[T], \lambda_3^A[T]), \dots$$

the trivial solution of (3.1.3) is unstable.

The trivial solution of (3.1.3) is stable for $\lambda = \lambda_{2k-1}^P[T]$ or $\lambda = \lambda_{2k}^P[T]$, $k = 1, 2, \dots$, if and only if $\lambda_{2k-1}^P[T] = \lambda_{2k}^P[T]$. Analogously, such solution is stable for $\lambda = \lambda_{2k}^A[T]$ or $\lambda = \lambda_{2k+1}^A[T]$, $k = 0, 1, \dots$, if and only if $\lambda_{2k}^A[T] = \lambda_{2k+1}^A[T]$.

Graphically, the function $\Delta(\lambda)$ would have an appearance similar to Figure 3.1.1 (Figure 2.4 in [23]).


 Figure 3.1.1: Graphic of $\Delta(\lambda)$.

Finally, we will briefly summarize some spectral properties which will be used in this chapter.

In [158, Chapter 4], it is proved a general result that includes the following Sturm-Liouville equation:

$$-(pu')'(t) + q(t)u(t) = \lambda w(t)u(t), \quad \text{a.e. } t \in (a, b), \quad -\infty \leq a < b \leq \infty, \quad (3.1.4)$$

with p, q and w real valued functions such that $1/p, q, w \in L^1((a, b))$ and $w > 0$ a.e. on (a, b) .

Moreover, the following two types of boundary conditions are considered: coupled self-adjoint boundary conditions, namely

$$u(b) = k_{11}u(a) + k_{12}(pu')(a), \quad (pu')(b) = k_{21}u(a) + k_{22}(pu')(a), \quad (3.1.5)$$

with

$$k_{11}k_{22} - k_{21}k_{12} = 1; \quad (3.1.6)$$

and separated self-adjoint boundary conditions:

$$A_1u(a) + A_2(pu')(a) = 0, \quad B_1u(b) + B_2(pu')(b) = 0, \quad (3.1.7)$$

with $A_1, A_2, B_1, B_2 \in \mathbb{R}$ satisfying $(A_1, A_2) \neq (0, 0)$ and $(B_1, B_2) \neq (0, 0)$.

Thus, in [158, Theorem 4.3.1] the following results are proved:

1. All the eigenvalues of problem (3.1.4) related to boundary conditions coupled (3.1.5) or separated (3.1.7), are real, isolated with no finite accumulation point, and there is an infinite but countable number of them.

2. If $p > 0$ on (a, b) and the coupled boundary conditions (3.1.5) are fulfilled, then the eigenvalues are bounded from below and can be ordered to satisfy

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots; \quad \lambda_k \rightarrow \infty, \text{ as } k \rightarrow \infty \quad (3.1.8)$$

Each eigenvalue may be simple or double but there cannot be two consecutive equalities in (3.1.8) since, for any value of λ , equation (3.1.4) has exactly two linearly independent solutions. Note that λ_k is well defined for each $k \geq 0$ but there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue since every nontrivial solution of the equation for such an eigenvalue is an eigenfunction. Given such an indexing scheme, let u_k be a real-valued eigenfunction of λ_k for the coupled conditions (3.1.5), $k \geq 0$, then the number of zeros of u_k in (a, b) is 0 or 1, if $k = 0$, and $k - 1$ or k or $k + 1$ if $k \geq 1$.

3. If $p > 0$ and the boundary conditions are the separated ones (3.1.7) then strict inequality holds everywhere in (3.1.8). Furthermore, if u_k is an eigenfunction of λ_k , then u_k is unique up to constant multiples and has exactly k zeros in the open interval (a, b) .

It is important to point out that the coupled conditions (3.1.5) cover the periodic boundary conditions ($k_{11} = k_{22} = 1, k_{21} = k_{12} = 0$). In this case, if $a, b \in \mathbb{R}$, Krein-Rutman Theorem ensures that the least eigenvalue is simple with its corresponding eigenfunction strictly positive on (a, b) and that the rest of the eigenfunctions change its sign on (a, b) .

Note also that coupled conditions (3.1.5) cover also the antiperiodic boundary conditions ($k_{11} = k_{22} = -1, k_{21} = k_{12} = 0$). In this case, Krein-Rutman Theorem is not applicable (because the corresponding Green's function always changes its sign).

On the other hand, the separated conditions (3.1.7) cover Neumann, Dirichlet and mixed conditions.

3.2. Hill's Equation

As we have said before, in this section we will particularize all the results obtained in Chapter 2 to the particular case of considering Hill's operator defined in (3.0.1). This will be done in Subsections 3.2.3 and 3.2.4.

Furthermore, in Subsection 3.2.5 we will complete the study of Hill's operator by proving that the eigenvalues related to problems (N, T) , (D, T) , (M_1, T) and (M_2, T) satisfy a certain order relation.

Finally, in Subsection 3.2.6 we will use all the relations between different Green's functions to deduce some explicit criteria to ensure the constant sign of some Green's functions, as well as some upper bounds for the first eigenvalues.

All the results dealing with this particular case of considering Hill's equation are included in [22] and [23].

3.2.1. Historical Background and Applications

Hill's equation (which is named after the pioneering work of the mathematical astronomer George William Hill (1838–1914), see [76]) has numerous applications in engineering and physics. Among them we can find some problems in mechanics, astronomy, circuits, electric conductivity of metals and cyclotrons.

As a first example of the Hill's equation we could consider a mass-spring system, that is, a spring with a mass m hanging from it. It is very well-known that, denoting by $x(t)$ the position of the mass at the instant t and assuming absence of friction, the previous model can be expressed as

$$x''(t) + \frac{k}{m} x(t) = 0,$$

with $k > 0$ the elastic constant of the string.

However, in a real physical system, there exists a friction force which opposes the movement and is proportional to the object's speed. In this case the situation can be modelled by the equation

$$x''(t) + \mu x'(t) + \frac{k}{m} x(t) = 0,$$

with μ the so-called friction coefficient. The value of such coefficient is characteristic of the environment where the object oscillates, and depends, among other variables, on the density, temperature and pressure of the environment. However, it could be considered a situation in which the spring moves between two different environments, each one with its particular friction coefficient. Also, the environment could have strong variations of density or temperature that could cause changes in the friction coefficient depending on time. This could be modelled by substituting the friction coefficient μ for a not necessarily constant function $\mu(t)$

$$x''(t) + \mu(t) x'(t) + \frac{k}{m} x(t) = 0.$$

Another possible situation would be that one in which there exists another external force acting periodically on the mass in such a way that it tends to move the

mass back into its position of equilibrium, acting in proportion to the distance to that position. Including this new variable in the previous model we have

$$x''(t) + \mu(t) x'(t) + \left(\frac{k}{m} + F(t) \right) x(t) = 0.$$

In any of the two cases, we obtain an equation in the form (3.0.2) in which, if $\mu(t)$ has enough regularity, we could do the following change of variable

$$a(t) = \frac{k}{m} + F(t) - \frac{1}{4} \left(\frac{\mu(t)}{m} \right)^2 - \frac{1}{2} \frac{\mu'(t)}{m}$$

and transform the equation into one in the form (3.0.1).

A second example studied in [42, 98] is the inverted pendulum. A mathematical pendulum consists of a particle of mass m connected to a base through a string (which is supposed to be rigid and of despicable weight) in such a way that the mass moves in a fixed vertical plane. If the particle moves by the force of gravity, then the movement of the pendulum is given by the equation

$$\theta''(t) - \frac{g}{l} \sin(\theta(t)) = 0,$$

where g denotes the gravity, l the length of the string and θ represents the angle between the string and the perpendicular line to the base.

In the surroundings of the equilibrium point $\theta = 0$, we can approximate $\sin \theta \approx \theta$, so the equation of movement could be rewritten as

$$\theta''(t) - \frac{g}{l} \theta(t) = 0.$$

Consider now the case in which the suspension point of the string vibrates vertically with an acceleration $a(t)$. Then, as it is proved in [42], the equation of movement would change into

$$\theta''(t) - \frac{1}{l} (g + a(t)) \theta(t) = 0,$$

which is of the form (3.0.1).

Other equations which fit into the framework of the Hill's equation are the following ones:

- Airy's equation: (see [133])

$$u''(t) + t u(t) = 0.$$

This equation appears in the study of the diffraction of light, the diffraction of radio waves around the Earth's surface, in aerodynamics and in the swing of an uniform vertical column which bounds under its own weight.

- Mathieu's equation: (see [19, 142, 162])

$$u''(t) + (c + b \cos t) u(t) = 0.$$

It is the result of the analysis of the phenomenon of parametric resonance associated with an oscillator whose parameters change with time. It appears in problems related to periodic movements, as the trajectory of an electron in a periodic arrangement of atoms.

3.2.2. Preliminary Results

Hill's operator properties have been described in several papers, where existence and multiplicity results, comparison principles, Green's functions and spectral analysis were studied. Some of these results can be found in [20–22, 142, 161].

In particular, the periodic problem related to Hill's equation, namely

$$\begin{cases} Lu(t) = 0 & \text{a. e. } t \in I, \\ u(0) = u(T), u'(0) = u'(T), \end{cases} \quad (P, T)$$

has been widely studied (see [19, 21, 142, 161, 162] and references therein).

Next we compile some properties which are satisfied by the Green's function related to problem (P, T) , $G_P[T]$, and which will be basic tools to prove some of our results.

Notation 3.2.1. *Note that, as in Chapter 2, we will use the notation $G[T]$ to refer to the Green's function related to operator L .*

Moreover, analogously to what we have done in Section 1.1, we will consider the parametrized operators defined from L or \tilde{L} . In particular, we will denote by $L[\lambda] u(t) \equiv L u(t) + \lambda u(t)$. In this case, to stress also its dependence on λ , we will denote by $G[\lambda, T]$ the Green's function related to $L[\lambda]$. Analogous notation will be used for $\tilde{L}[\lambda] u(t) \equiv \tilde{L} u(t) + \lambda u(t)$.

Lemma 3.2.2. [21, Lemma 2.2] *Suppose that the Green's function $G_P[T]$ does not change sign on $I \times I$ and vanishes at some point $(t_0, s_0) \in I \times I$, then $t_0 = s_0$, $(t_0, s_0) = (0, T)$ or $(t_0, s_0) = (T, 0)$.*

Lemma 3.2.3. [21, Lemma 2.4] *If $G_P[T] \leq 0$ on $I \times I$ then $G_P[T] < 0$ on $I \times I$.*

Lemma 3.2.4. [161, Theorem 1.1] *Suppose that $a \in L^1(I)$, then:*

1. $G_P[T] < 0$ on $I \times I$ if and only if $\lambda_0^P[T] > 0$.
2. $G_P[T] \geq 0$ on $I \times I$ if and only if $\lambda_0^P[T] < 0 \leq \lambda_0^A[T]$.

By introducing the parametrized potentials $a + \lambda$, with $\lambda \in \mathbb{R}$, the previous result could be rewritten as follows.

Lemma 3.2.5. [161, Theorem 1.2] Suppose that $a \in L^1(I)$, then:

1. $G_P[\lambda, T] < 0$ on $I \times I$ if and only if $\lambda < \lambda_0^P[T]$.
2. $G_P[\lambda, T] \geq 0$ on $I \times I$ if and only if $\lambda_0^P[T] < \lambda \leq \lambda_0^A[T]$.

To finish with these preliminary results, we include the following property which is satisfied by Green's functions related to Hill's operator coupled with any of the boundary conditions considered in this chapter.

Lemma 3.2.6. [21, Lemma 2.8] Let λ_1, λ_2 be such that the Green's functions of the corresponding problem, $G[\lambda_1, T]$ and $G[\lambda_2, T]$, have the same constant sign on $I \times I$. If $\lambda_1 > \lambda_2$ then $G[\lambda_1, T](t, s) < G[\lambda_2, T](t, s)$ for all $(t, s) \in I \times I$.

3.2.3. Constant Sign of Green's Functions

In this subsection we will study the constant sign of the Green's functions of various of the boundary value problems which have been previously considered (namely, Neumann, Dirichlet, mixed and periodic).

The results in this subsection complement those in Section 2.4 for the particular case of Hill's equation.

First, we will prove a necessary condition that must be satisfied by the Green's function of a self-adjoint operator. This result generalizes the one obtained for the periodic case in Lemma 3.2.2 and it is valid for periodic, Neumann and Dirichlet problems.

Proposition 3.2.7. [22, Proposition 3.1] Assume that operator L is nonresonant and self-adjoint on a Banach space X . If the Green's function $G[T]$ does not change sign on $I \times I$ and $G[T]$ vanishes at some point $(t_0, s_0) \in I \times I$, then either (t_0, s_0) belongs to the diagonal of the square $I \times I$ or (t_0, s_0) is in the boundary of $I \times I$, that is, at least one of the three following properties hold:

1. $t_0 = s_0 \in I$.
2. $t_0 = 0$ or $t_0 = T$.
3. $s_0 = 0$ or $s_0 = T$.

Proof. Suppose, on the contrary, that $G[T](t_0, s_0) = 0$ with $(t_0, s_0) \in (0, T) \times (0, T)$ such that $t_0 \neq s_0$. Since $G[T](t_0, s_0) = G[T](s_0, t_0)$, we may assume that $t_0 > s_0$.

By definition of the Green's function, we know that

$$x(t) \equiv G[T](t, s_0), \quad t \in I,$$

solves the equation

$$\begin{cases} x''(t) + a(t)x(t) = 0, & \text{a.e. } t \in (s_0, T], \\ x(t_0) = x'(t_0) = 0. \end{cases}$$

Then, $G[T](t, s_0) = 0$ for all $t \in (s_0, T]$ and, in consequence, from the symmetric property, $G[T](s_0, s) = 0$ for all $s \in (s_0, T]$.

Now, fix $s \in (s_0, T]$. Since $G[T]$ is nonnegative on $I \times I$, we have that function

$$y(t) \equiv G[T](t, s), \quad t \in I,$$

is a solution of

$$\begin{cases} y''(t) + a(t)y(t) = 0, & \text{a.e. } t \in [0, s), \\ y(s_0) = y'(s_0) = 0. \end{cases}$$

Once again, $G[T](t, s) = 0$ for all $s \in (s_0, T]$ and all $t \in [0, s)$.

From symmetry, we deduce $G[T](t, s) = 0$ for all $t \in (s_0, T]$ and $s \in [0, t)$. This contradicts property (G3) in the definition of the Green's function (Definition 1.1.1) and so we deduce the result. \square

Remark 3.2.8. Note that in the proof of previous proposition we use the uniqueness of solution of the initial boundary value problem to conclude that a nontrivial solution of a differential equation of order 2 can never have a zero of multiplicity two. Obviously, this does not remain true for differential equations of order higher than two and this is the reason why previous result is not applicable to the general case of the $2n$ -th order operator.

Remark 3.2.9. If we consider the periodic case with $a(t) = \left(\frac{\pi}{T}\right)^2$, using [24] we obtain the following expression for the Green's function

$$G_P[T](t, s) = \frac{T}{2\pi} \begin{cases} \sin\left(\frac{\pi(t-s)}{T}\right), & 0 \leq s \leq t \leq T, \\ \sin\left(\frac{\pi(t-s+T)}{T}\right), & 0 \leq t < s \leq T, \end{cases}$$

which is strictly positive on $I \times I$ except for the diagonal and the points $(0, T)$ and $(T, 0)$.

On the other hand, when $a(t) = k^2 < \left(\frac{\pi}{T}\right)^2$ and the Dirichlet boundary conditions are studied, we have that the Green's function is given by the following expression

$$G_D[T](t, s) = \frac{1}{k \sin(kT)} \begin{cases} \sin(ks) \sin(k(t-T)), & 0 \leq s \leq t \leq T, \\ \sin(kt) \sin(k(s-T)), & 0 \leq t < s \leq T. \end{cases}$$

We observe that $G_D[T]$ is strictly negative on $(0, T) \times (0, T)$ and vanishes on the boundary of its square of definition.

In consequence, the previous result cannot be improved for general self-adjoint Hill's operators.

In particular, if Neumann boundary conditions are considered, we obtain a more precise localization of the zeros of the related Green's function.

Lemma 3.2.10. [22, Lemma 4.1] Suppose that the Green's function $G_N[T]$ is non-negative on $I \times I$ and there is some $(t_0, s_0) \in I \times I$ for which $G_N[T](t_0, s_0) = 0$, then either $(t_0, s_0) = (0, 0)$ or $(t_0, s_0) = (T, T)$.

Proof. Suppose that $G_N[T](t_0, s_0) = 0$ for some $(t_0, s_0) \in I \times I$. Since $G_N[T] \geq 0$ on $I \times I$, as operator L is self-adjoint, Proposition 3.2.7 lets us conclude that (t_0, s_0) belongs either to the boundary of the square of definition or to its diagonal.

In the first case, suppose that $t_0 \in (0, T)$ and $s_0 = 0$. Then we have that $x_0(t) \equiv G_N[T](t, 0)$ satisfies the equation

$$\begin{cases} x_0''(t) + a(t) x_0(t) = 0, & t \in (0, T], \\ x_0(t_0) = x_0'(t_0) = 0, \end{cases}$$

which means that $G_N[T](t, 0) \equiv 0$ on $(0, T]$.

From the symmetry of $G_N[T]$, we have that $G_N[T](0, s) \equiv 0$ for all $s \in (0, T]$.

As a consequence, $x_s(t) \equiv G_N[T](t, s)$ satisfies the equation

$$\begin{cases} x_s''(t) + a(t) x_s(t) = 0, & t \in [0, s), \\ x_s(0) = x_s'(0) = 0, \end{cases}$$

which implies that $G_N[T](t, s) \equiv 0$ for all $t < s$. Using again the symmetry of $G_N[T]$ we have that it is identically zero on $I \times I$ and we reach a contradiction.

Previous argument is valid for all (t_0, s_0) in the boundary of $I \times I$ except for $(0, 0)$ and (T, T) .

Assume now that $G_N[T](t_0, t_0) = 0$ for some $t_0 \in (0, T)$. In this case, defining $x_{t_0}(t)$ as the even extension to J of $G_N[T](t, t_0)$, we have that it satisfies the equation

$$\begin{cases} x_{t_0}''(t) + \tilde{a}(t) x_{t_0}(t) = 0, & t \in (t_0, 2T - t_0), \\ x_{t_0}(t_0) = x_{t_0}(2T - t_0) = 0, \end{cases}$$

where, as usual, \tilde{a} denotes the even extension of a to the interval J .

From Sturm's comparison Theorem (Theorem 3.1.3), we have that for any $\lambda \geq 0$ every nontrivial solution of the equation

$$y''(t) + (\tilde{a}(t) + \lambda) y(t) = 0, \quad t \in [0, 2T], \quad (3.2.1)$$

has at least one zero on $[t_0, 2T - t_0]$.

Now, note that the even extension to J of the positive eigenfunction on $(0, T]$ associated with $\lambda_0^{M_2}[T]$ solves (3.2.1) but does not have any zero on $[t_0, 2T - t_0]$. Therefore we deduce that $\lambda_0^{M_2}[T] < 0$. Furthermore, note that the aforementioned extension is positive on $(0, 2T)$ and cancels both at 0 and at $2T$.

As a consequence, for any $\lambda \in (\lambda_0^{M_2}[T], 0]$ we have that y_0 , the even extension to J of $G_N[\lambda, T](t, 0)$, has at least one zero on $(0, 2T)$. Moreover, all the zeros of y_0 are simple because otherwise $G_N[\lambda, T](t, 0) \equiv 0$ on $(0, T]$, which cannot happen. Then necessarily y_0 changes its sign on $(0, 2T)$ and, as it is an even function, $G_N[\lambda, T](t, 0)$ changes its sign on $(0, T)$. This contradicts the hypothesis that $G_N[T]$ is nonnegative on $I \times I$.

This way, we conclude that $G_N[T]$ can only vanish at $(0, 0)$ or (T, T) . \square

Remark 3.2.11. Note that if $G_N[T](0, 0) = 0$ we have that $x(t) \equiv G_N[T](t, 0)$ is a solution of

$$\begin{cases} x''(t) + a(t)x(t) = 0, & t \in I, \\ x(0) = x'(T) = 0. \end{cases} \quad (3.2.2)$$

Moreover, when $G_N[T](T, T) = 0$, $y(t) = G_N[T](t, T)$ is a solution of

$$\begin{cases} y''(t) + a(t)y(t) = 0, & t \in I, \\ y'(0) = y(T) = 0. \end{cases} \quad (3.2.3)$$

As a consequence of previous result and equality (2.2.1), we deduce the following corollary.

Corollary 3.2.12. If $G_P[2T]$ has constant sign on $J \times J$, then it holds that $G_N[T]$ has the same sign as $G_P[2T]$ on $I \times I$. In such a case, $G_N[T](t, s)$ is different from zero for all $(t, s) \in (I \times I) \setminus \{(0, 0) \cup (T, T)\}$.

Moreover, $G_N[T](0, 0) = 0$ if and only if equation (3.2.2) has a non zero and constant sign solution on $[0, T]$, which means that $\lambda_0^{M_2}[T] = 0$.

$G_N[T](T, T) = 0$ if and only if equation (3.2.3) has a non zero and constant sign solution on $(0, T]$, which means that $\lambda_0^{M_1}[T] = 0$.

Reasoning as in Lemma 3.2.10, it is deduced the following.

Lemma 3.2.13. *Suppose that $a \in L^1(I)$ and the Green's function $G_D[T]$ has constant sign on $I \times I$ and there exists some $(t_0, s_0) \in I \times I$ such that $G_D[T](t_0, s_0) = 0$. Then (t_0, s_0) belongs to the boundary of the square of definition of $G_D[T]$.*

Remark 3.2.14. *From the Dirichlet boundary conditions and property (G5) of the Green's function (see Definition 1.1.1), it is clear that $G_D[T]$ must cancel on the whole boundary of $I \times I$. Previous lemma ensures that, when $G_D[T]$ has constant sign, it can not vanish at any other point.*

In the sequel we will prove that $G_D[T]$ can never be nonnegative when working with Hill's equation and it is negative on $(0, T) \times (0, T)$ for λ smaller than the first eigenvalue.

Lemma 3.2.15. [23, Lemma 36] *Suppose that $a \in L^1(I)$, then:*

$G_D[\lambda, T] < 0$ on $(0, T) \times (0, T)$ if and only if $\lambda < \lambda_0^D[T]$.

Moreover, if $\lambda > \lambda_0^D[T]$ is such that $G_D[\lambda, T]$ exists, then $G_D[\lambda, T]$ changes sign on $I \times I$.

Proof. Choose $\lambda < \lambda_0^D[T]$. From Theorem 3.1.3 it is clear that any solution of equation

$$u''(t) + (a(t) + \lambda) u(t) = 0, \quad t \in I, \quad (3.2.4)$$

has at most one zero on I .

From Definition 1.1.1, it holds that for each $s_0 \in (0, T)$, $u_{s_0}(\cdot) \equiv G_D[\lambda, T](\cdot, s_0)$ satisfies (3.2.4) on $[0, s_0] \cup (s_0, T]$.

Then, if u is the unique solution of (3.2.4) under the initial conditions

$$u(0) = 0, \quad u'(0) = 1,$$

it is clear that there exists a constant k_1 such that $u_{s_0}(t) = k_1 u(t)$ for all $t < s_0$. Obviously, this constant k_1 depends on the value s_0 considered, so we could say that there exists a function (which, for the sake of simplicity, will be denoted also by k_1) such that

$$G_D[\lambda, T](t, s) = k_1(s) u(t) \quad \text{for all } t < s.$$

Moreover, since $u(0) = 0$, we have that $u(t) \neq 0$ for all $t \in (0, T]$.

Analogously, if v is the unique solution of (3.2.4) satisfying the final conditions

$$u(T) = 0, \quad u'(T) = -1,$$

then there exists a constant k_2 such that $u_{s_0}(t) = k_2 v(t)$ for all $t > s_0$. Consequently,

$$G_D[\lambda, T](t, s) = k_2(s) v(t) \quad \text{for all } s < t.$$

In this case, $v(T) = 0$ implies that $v(t) \neq 0$ for all $t \in [0, T)$.

Now, since $G_D[\lambda, T]$ is a symmetric function, necessarily $k_1(s) = cv(s)$ and $k_2(s) = cu(s)$ for some non zero constant c , that is,

$$G_D[\lambda, T](t, s) = \begin{cases} cv(s)u(t), & 0 \leq t < s \leq T, \\ cu(s)v(t), & 0 \leq s < t \leq T, \end{cases}$$

and, since $G_D[\lambda, T]$ is continuous on $I \times I$, it is clear that

$$G_D[\lambda, T](s, s) = cu(s)v(s).$$

Therefore $G_D[\lambda, T]$ has strict constant sign on $(0, T) \times (0, T)$ for all $\lambda < \lambda_0^D[T]$. We will see now that this sign has to be necessarily negative.

On the contrary, assume that there exists some value $\bar{\lambda} < \lambda_0^D[T]$ for which $G_D[\bar{\lambda}, T] > 0$ on $(0, T) \times (0, T)$. From this property, since

$$\frac{\partial G[\bar{\lambda}, T]}{\partial t}(0, s) \neq 0 \quad \text{and} \quad \frac{\partial G[\bar{\lambda}, T]}{\partial t}(T, s) \neq 0, \quad \text{for all } s \in (0, T),$$

it is immediate to verify that, choosing $\phi(t) = t(T - t)$, for all $s \in (0, T)$ we have that

$$\kappa_1(s) = \min_{t \in I} \frac{G_D[\bar{\lambda}, T](t, s)}{\phi(t)} \in (0, \infty)$$

and

$$\kappa_2(s) = \max_{t \in I} \frac{G_D[\bar{\lambda}, T](t, s)}{\phi(t)} \in (0, \infty)$$

and are continuous functions on I .

Then property (P_g) in Lemma 1.1.9 is fulfilled. Thus, a necessary condition for $G_D[\bar{\lambda} + \mu, T]$ to be nonnegative on $I \times I$ is that $\mu > \lambda_0^D[\bar{\lambda}, T]$, being $\lambda_0^D[\bar{\lambda}, T]$ the smallest eigenvalue of operator $L[\bar{\lambda}]$ coupled with Dirichlet conditions.

Now, taking into account that

$$\lambda_0^D[\bar{\lambda}, T] = \lambda_0^D[T] - \bar{\lambda},$$

we have that a necessary condition for $G_D[\bar{\lambda} + \mu, T]$ to be nonnegative on $I \times I$ is that $\bar{\lambda} + \mu > \lambda_0^D[T]$ or, which is the same, if $G_D[\lambda, T] \geq 0$ on $I \times I$ then $\lambda > \lambda_0^D[T]$. This facts contradicts the existence of such $\bar{\lambda}$.

As a consequence, $G_D[\lambda, T] < 0$ on $(0, T) \times (0, T)$ for all $\lambda < \lambda_0^D[T]$ and condition (N_g) is fulfilled.

Thus, from Lemma 1.1.8, we can ensure that $G_D[\lambda, T] < 0$ on $(0, T) \times (0, T)$ if and only if $\lambda < \lambda_0^D[T]$.

Now we will see that for $\lambda > \lambda_0^D[T]$ such that the Green's function $G_D[\lambda, T]$ exists, it holds that $G_D[\lambda, T]$ changes sign.

Take $\lambda_0^D[T] < \lambda < \lambda_1^D[T]$. From the results given at the end of Section 3.1, it is clear that the eigenfunction related to $\lambda_0^D[T]$ cancels at 0 and T but has strict constant sign on $(0, T)$, and that the eigenfunction related to $\lambda_1^D[T]$ cancels at 0 and T and has exactly one more zero on $(0, T)$. Taking this into account and using Sturm's comparison Theorem (Theorem 3.1.3), it is easy to deduce that any solution of equation

$$u''(t) + (a(t) + \lambda) u(t) = 0, \quad t \in I$$

has exactly one zero on $(0, T)$.

In particular, both u and v , defined in the first part of this proof, have exactly one zero on $(0, T)$. Let's denote by t_0 and t_1 the zeros of u and v , respectively, and assume that $t_1 < t_0$ (being the other case analogous).

Now, for $s < t_1$, $u(s) > 0$ and $v(s) < 0$, whereas for $t_1 < r < t_0$, $u(r) > 0$ and $v(r) > 0$. This way we have found two points $(s, s), (r, r) \in (0, T) \times (0, T)$ such that $G_D[\lambda, T](s, s) G_D[\lambda, T](r, r) < 0$.

As a consequence, for $\lambda_0^D[T] < \lambda < \lambda_1^D[T]$, $G_D[\lambda, T]$ changes sign and so from Lemma 1.1.9 we conclude that $G_D[\lambda, T]$ must change sign for every $\lambda > \lambda_0^D[T]$. \square

Remark 3.2.16. *The previous result is a particular case of [39, Theorem 3.1], where a characterization of the constant sign of the Green's function is proved for the general n -th order linear operator coupled with the so-called $(k, n - k)$ boundary conditions.*

It can be also deduced as a consequence of [40, Theorem 11], where it is shown that if a linear equation is disconjugate then the related Green's function has constant sign, and [38, Theorem 2.1], where the interval of disconjugation is characterized by means of the eigenvalues of some suitable boundary condition.

Remark 3.2.17. *It is important to note that Lemma 3.2.15 does not remain true when $n > 1$. Indeed, we have considered in Example 2.4.8 a fourth-order problem for which the Green's function of the Dirichlet problem is nonnegative.*

In the sequel, we characterize the intervals on which $G_N[\lambda, T]$ has constant sign.

Theorem 3.2.18. *The following equalities are fulfilled for any $a \in L^1(I)$.*

1. $G_N[\lambda, T] < 0$ on $I \times I$ if and only if $\lambda < \lambda_0^N[T] (= \lambda_0^P[2T])$.
2. $G_N[\lambda, T] > 0$ on $I \times I$ if and only if

$$(\lambda_0^P[2T] =) \lambda_0^N[T] < \lambda < \min \left\{ \lambda_0^{M_1}[T], \lambda_0^{M_2}[T] \right\}.$$

3. $G_N[\lambda, T](t, s) > 0$ for all $(t, s) \in (I \times I) \setminus \{(0, 0) \cup (T, T)\}$ if and only if

$$(\lambda_0^P[2T] =) \lambda_0^N[T] < \lambda \leq \min \left\{ \lambda_0^{M_1}[T], \lambda_0^{M_2}[T] \right\}.$$

4. $G_N[\lambda, T](0, 0) = 2 G_P[\lambda, 2T](0, 0)$ and $\lambda_0^{M_2}[T]$ is characterized as the first root of equation

$$(G_N[\lambda, T](0, 0) =) G_P[\lambda, 2T](0, 0) = 0.$$

5. $G_N[\lambda, T](T, T) = 2 G_P[\lambda, 2T](T, T)$ and $\lambda_0^{M_1}[T]$ is characterized as the first root of equation

$$(G_N[\lambda, T](T, T) =) G_P[\lambda, 2T](T, T) = 0.$$

Proof. From Lemmas 3.2.3 and 3.2.5, we know that $G_P[\lambda, 2T]$ is strictly negative on $J \times J$ if and only if $\lambda < \lambda_0^P[2T]$. Thus, equation (2.2.1) and the equality $\lambda_0^P[2T] = \lambda_0^N[T]$ (see Theorem 2.3.2), imply Assertion 1.

Moreover, from Lemmas 3.2.3 and 3.2.5, we know that $G_P[\lambda, 2T]$ is nonnegative on $J \times J$ if and only if $\lambda_0^P[2T] < \lambda \leq \lambda_0^A[2T]$. Then, (2.2.1) and the fact that $\lambda_0^P[2T] = \lambda_0^N[T]$ and $\lambda_0^A[2T] = \min \{\lambda_0^{M_1}[T], \lambda_0^{M_2}[T]\}$ (see Theorem 2.3.2), together with Lemma 3.2.10 and Corollary 3.2.12, imply Assertions 2 and 3.

Now, taking into account equality (2.2.1) and the fact that

$$G_P[\lambda, 2T](0, 0) = G_P[\lambda, 2T](2T, 0),$$

we conclude that for all $\lambda \in \mathbb{R}$ the following equalities hold

$$G_N[\lambda, T](0, 0) = 2 G_P[\lambda, 2T](0, 0), \quad (3.2.5)$$

and

$$G_N[\lambda, T](T, T) = 2 G_P[\lambda, 2T](T, T). \quad (3.2.6)$$

From Lemma 3.2.6, we have that while both values on equations (3.2.5) and (3.2.6) are positive, they are strictly decreasing with respect to λ . Thus, Corollary 3.2.12 ensures that $\lambda_0^{M_2}[T]$ is the first zero of (3.2.5) and $\lambda_0^{M_1}[T]$ the first zero of (3.2.6). Then, Assertions 4 and 5 hold. \square

For an arbitrary potential a we obtain the following corollaries.

Corollary 3.2.19. *The following equalities are fulfilled for any $a \in L^1(I)$.*

1. $G_N[T](t, s) < 0$ for all $(t, s) \in I \times I$ if and only if $0 < \lambda_0^P[2T]$ ($= \lambda_0^N[T]$).
2. $G_N[T](t, s) > 0$ for all $(t, s) \in (0, T) \times (0, T)$ if and only if

$$(\lambda_0^N[T] =) \lambda_0^P[2T] < 0 \leq \min \{\lambda_0^{M_1}[T], \lambda_0^{M_2}[T]\} \quad (\leq \lambda_0^D[2T] < \lambda_0^D[T]).$$

Corollary 3.2.20. *The following property holds for any $a \in L^1(I)$:*

$G_{M_2}[\lambda, T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$ if and only if $\lambda < \lambda_0^{M_2}[T]$.

Proof. From Lemma 3.2.15, we know that the Green's function $G_D[\lambda, 2T]$ is strictly negative on $(0, 2T) \times (0, 2T)$ if and only if $\lambda < \lambda_0^D[2T]$.

Considering then equation and the fact that, as we have seen in Theorem 2.3.2, $\lambda_0^D[2T] = \lambda_0^{M_2}[T]$, we conclude the result. \square

Moreover, as an immediate consequence of (2.2.16) and Corollary 3.2.20, we have the following characterization.

Corollary 3.2.21. *Let $a \in L^1(I)$, then:*

$G_{M_1}[\lambda, T](t, s) < 0$ for all $(t, s) \in [0, T] \times [0, T]$ if and only if $\lambda < \lambda_0^{M_1}[T]$.

From Theorems 2.3.2 and 3.2.18 and Corollaries 3.2.19, 3.2.20 and 3.2.21, we deduce some relations between the constant sign of various Green's functions.

Theorem 3.2.22. [22] *For any $a \in L^1(I)$ the following properties hold:*

1. $G_P[2T] < 0$ on $J \times J$ if and only if $G_N[T] < 0$ on $I \times I$. This is equivalent to $G_N[2T] < 0$ on $J \times J$.
2. $G_P[2T] > 0$ on $(0, 2T) \times (0, 2T)$ if and only if $G_N[T] > 0$ on $(0, T) \times (0, T)$.
3. If $G_N[2T] > 0$ on $(0, 2T) \times (0, 2T)$ then $G_N[T] > 0$ on $(0, T) \times (0, T)$.
4. If $G_P[2T] < 0$ on $J \times J$ then $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$.
5. If $G_P[2T] > 0$ on $(0, 2T) \times (0, 2T)$ then $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$.
6. If $G_N[T]$ (or, equivalently, $G_P[2T]$) has constant sign on $I \times I$, then $G_D[T] < 0$ on $(0, T) \times (0, T)$, $G_{M_1}[T] < 0$ on $[0, T] \times [0, T]$ and $G_{M_2}[T] < 0$ on $(0, T) \times (0, T)$.
7. $G_D[2T] < 0$ on $(0, 2T) \times (0, 2T)$ if and only if $G_{M_2}[T] < 0$ on $(0, T) \times (0, T)$.
8. If either $G_{M_2}[T] < 0$ on $(0, T) \times (0, T)$ or $G_{M_1}[T] < 0$ on $[0, T] \times [0, T]$, then $G_D[T] < 0$ on $(0, T) \times (0, T)$.

Remark 3.2.23. *Regarding Assertions 1 and 2, we recall that, for $n > 1$, the constant sign of $G_P[2T]$ implies the (same) constant sign of $G_N[T]$ and the equivalence holds when considering constant coefficients. Similarly, the negative sign of $G_N[2T]$ implies the negative sign of $G_N[T]$ but the equivalence does not hold, not even for the constant case.*

Assertion 3 in previous theorem coincides with Assertion 4 in Corollary 2.4.1.

Finally, with respect to Assertion 7, for $n > 1$, the negative sign of $G_D[2T]$ implies the negative sign of $G_{M_2}[T]$ but the converse does not hold, not even for constant coefficients.

Remark 3.2.24. Note that Assertion 6 in Corollary 2.4.1 has not been included in previous result. The reason is that, as we have seen in Lemma 3.2.15, for $n = 1$, the Green's function related to $(D, 2T)$ can never be nonnegative.

We will show now some counterexamples in which we will see that Assertions 4, 5, 6 and 8 do not hold, in general, for $n > 1$.

Next example shows that Assertions 4 and 5 in Theorem 3.2.22 are not true in general.

Example 3.2.25. Consider the periodic and Dirichlet problems on the same interval $[0, 2T] = [0, 3]$ related to operator

$$\tilde{L}u(t) \equiv u^{(4)}(t) + (t(t-3) + \lambda)u(t), \quad t \in [0, 3]. \quad (3.2.7)$$

By numerical approach, we have obtained that for $\lambda = -1.5$, $G_P[2T]$ is negative while $G_D[2T]$ changes its sign on $J \times J$.

Moreover, for $\lambda = 15$, $G_P[2T]$ is positive while $G_D[2T]$ changes sign again.

We will see in the two following examples that none of the implications given in Assertion 6 in Theorem 3.2.22 holds for $n > 1$.

Example 3.2.26. Consider now $[0, T] = [0, 2]$ and operators L and \tilde{L} given in (2.4.1) and (2.4.2).

For $\lambda = -2$, one can check that both $G_P[2T]$ and $G_N[T]$ are nonpositive, whereas $G_D[T]$ and $G_{M_1}[T]$ are nonnegative.

For $\lambda = 2$, it occurs that both $G_P[2T]$ and $G_N[T]$ are nonnegative, whereas $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ are nonnegative.

Example 3.2.27. Take now $[0, T] = [0, \frac{3}{2}]$, the operator L given by

$$Lu(t) \equiv u^{(4)}(t) + (t(t-3) + \lambda), \quad t \in \left[0, \frac{3}{2}\right]$$

and operator \tilde{L} given in (3.2.7).

In this case, for $\lambda = 1.5$, it occurs that $G_P[2T]$ and $G_N[T]$ are nonpositive, whereas $G_{M_2}[T]$ is nonnegative.

Finally, we will show that Assertion 8 in Theorem 3.2.22 does not hold either when $n > 1$.

Example 3.2.28. Consider again $[0, T] = [0, 2]$ and operators L and \tilde{L} given in (2.4.1) and (2.4.2).

In this case, for $\lambda = -6$, $G_{M_1}[T]$ is nonpositive but $G_D[T]$ is nonnegative.

Similarly, for $\lambda = -2$, $G_{M_2}[T]$ is nonpositive but $G_D[T]$ is nonnegative.

3.2.4. Comparison Principles

In this subsection we complement the results proved in Section 2.5 for $n > 1$.

As we have seen in previous section, for the particular case of Hill's equation, we have stronger relations between the constant sign of various Green's functions than for the general $2n$ -th order operator. Therefore, we can obtain stronger comparison principles too.

On the other hand, note that some results which have been proved for the general even order problem do not make sense for Hill's equation. This is the case of Item 4 in Corollary 2.5.3 or Item 1 in Theorem 2.5.5, which do not make sense since, as we have seen, $G_D[2T]$ related to Hill's equation can never be nonnegative on $J \times J$.

Now, we will see how to adapt the results in Section 2.5 to this case.

In particular, from Theorem 3.2.22, results in Corollary 2.5.1 can be improved for this case in the following way.

Corollary 3.2.29. If $G_P[2T] \geq 0$ on $J \times J$, then

$$G_N[T](t, s) \geq -G_D[T](t, s) \geq 0, \quad \forall (t, s) \in I \times I.$$

If $G_P[2T] < 0$ on $J \times J$, then

$$G_N[T](t, s) < G_D[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I.$$

As a consequence, Theorem 2.5.2 can also be rewritten in such a way that we may assure that both solutions have constant sign.

Theorem 3.2.30. Let u_N be the unique solution of problem (N, T) for $\sigma = \sigma_1$ and u_D the unique solution of problem (D, T) for $\sigma = \sigma_2$. Then

1. If $G_P[2T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a. e. $t \in I$, then

$$|u_D(t)| \leq u_N(t) \quad \text{for all } t \in I.$$

1.1 If, moreover, $\sigma_2(t) \geq 0$ a. e. $t \in I$, then

$$-u_N(t) \leq u_D(t) \leq 0 \quad \text{for all } t \in I.$$

1.2 If, moreover, $\sigma_2(t) \leq 0$ a. e. $t \in I$, then

$$0 \leq u_D(t) \leq u_N(t) \quad \text{for all } t \in I.$$

2. If $G_P[2T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then

$$u_N(t) \leq u_D(t) \leq 0 \quad \text{for all } t \in I.$$

3. If $G_P[2T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then

$$0 \leq u_D(t) \leq u_N(t) \quad \text{for all } t \in I.$$

This situation is represented in Figures 3.2.1, 3.2.2, 3.2.3 and 3.2.4.

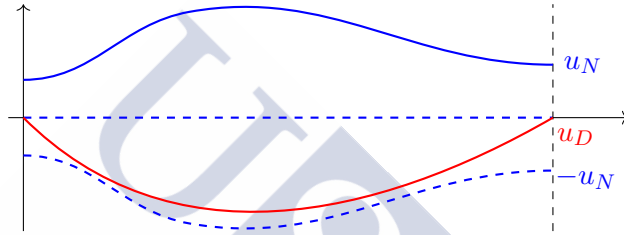


Figure 3.2.1: Solutions of (N, T) and (D, T) in Case 1.1 in Theorem 3.2.30.

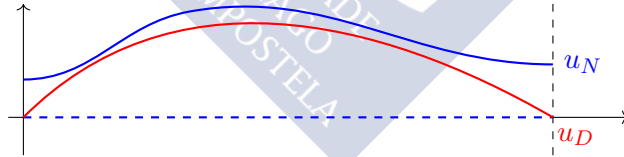


Figure 3.2.2: Solutions of (N, T) and (D, T) in Case 1.2 in Theorem 3.2.30.

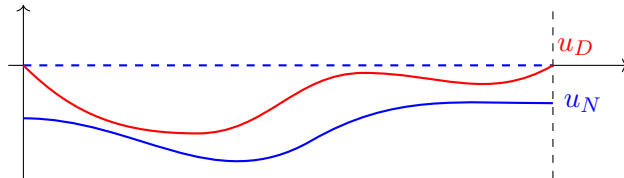


Figure 3.2.3: Solutions of (N, T) and (D, T) in Case 2 in Theorem 3.2.30.

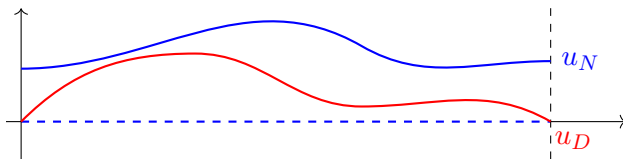


Figure 3.2.4: Solutions of (N, T) and (D, T) in Case 3 in Theorem 3.2.30.

Remark 3.2.31. Note that Theorem 3.2.30 is stronger than its corresponding one for the general even order equation (namely, Theorem 2.5.2) as for Hill's equation we are able to ensure the constant sign of both Neumann and Dirichlet solutions, which did not happen in Theorem 2.5.2.

Analogously, Corollary 2.5.3 can be improved in the following way.

Corollary 3.2.32. If $G_N[2T] \geq 0$ on $J \times J$, then

$$G_N[T](t, s) \geq -G_{M_1}[T](t, s) \geq 0, \quad \forall (t, s) \in I \times I.$$

If $G_N[2T] < 0$ on $J \times J$, then

$$G_N[T](t, s) < G_{M_1}[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I.$$

If $G_D[2T] \leq 0$ on $J \times J$, then

$$G_{M_2}[T](t, s) < G_D[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I.$$

Remark 3.2.33. Note that Assertion 4 in Corollary 2.5.3 has not been included in previous result since, as we have seen in Lemma 3.2.15, for Hill's equation, the Green's function related to $(D, 2T)$ can never be nonnegative.

Now, we can adapt Theorems 2.5.4 and 2.5.5 for the case of Hill's equation.

Theorem 3.2.34. Let u_N be the unique solution of problem (N, T) for $\sigma = \sigma_1$ and u_{M_1} the unique solution of (M_1, T) for $\sigma = \sigma_2$. Then

1. If $G_N[2T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a. e. $t \in I$, then

$$|u_{M_1}(t)| \leq u_N(t) \quad \text{for all } t \in I.$$

1.1 If, moreover, $\sigma_2(t) \geq 0$ a. e. $t \in I$, then

$$-u_N(t) \leq u_{M_1}(t) \leq 0 \quad \text{for all } t \in I.$$

1.2 If, moreover, $\sigma_2(t) \leq 0$ a. e. $t \in I$, then

$$0 \leq u_{M_1}(t) \leq u_N(t) \quad \text{for all } t \in I.$$

2. If $G_N[2T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then

$$u_N(t) \leq u_{M_1}(t) \leq 0 \quad \text{for all } t \in I.$$

3. If $G_N[2T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then

$$0 \leq u_{M_1}(t) \leq u_N(t) \quad \text{for all } t \in I.$$

Theorem 3.2.35. Suppose that $G_D[2T] \leq 0$ on $J \times J$. Let u_{M_2} be the unique solution of problem (M_2, T) for $\sigma = \sigma_1$ and u_D the unique solution of problem (D, T) for $\sigma = \sigma_2$.

1. If $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a. e. $t \in I$, then

$$u_{M_2}(t) \leq u_D(t) \leq 0 \quad \text{for all } t \in I.$$

2. If $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a. e. $t \in I$, then

$$0 \leq u_D(t) \leq u_{M_2}(t) \quad \text{for all } t \in I.$$

Remark 3.2.36. We note that, since Assertion 4 in Corollary 2.5.3 can never happen for the case studied in this section, it implies that hypotheses in Assertion 1 in Theorem 2.5.5 are never fulfilled in such a case. Therefore, we have not included the corresponding Assertion in Theorem 3.2.35.

Moreover, using the characterization given in Subsection 3.2.3, it is possible to rewrite Corollaries 3.2.29 and 3.2.32 in terms of eigenvalues, as follows.

Corollary 3.2.37. If $(\lambda_N[T] =) \lambda_P[2T] < 0 \leq \lambda_A[2T]$, then

$$G_N[T](t, s) \geq -G_D[T](t, s) \geq 0, \quad \forall (t, s) \in I \times I.$$

If $(\lambda_N[T] = \lambda_N[2T] = \lambda_P[4T] =) \lambda_P[2T] > 0$, then

$$G_N[T](t, s) < G_D[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I$$

and

$$G_N[T](t, s) < G_{M_1}[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I.$$

If $(\lambda_N[T] = \lambda_N[2T] =) \lambda_P[4T] < 0 \leq \lambda_A[4T]$, then

$$G_N[T](t, s) \geq -G_{M_1}[T](t, s) \geq 0, \quad \forall (t, s) \in I \times I.$$

If $(\lambda_D[2T] =) \lambda_{M_2}[T] > 0$, then

$$G_{M_2}[T](t, s) < G_D[T](t, s) \leq 0, \quad \forall (t, s) \in I \times I.$$

Finally, we are also able to deduce the following result which is not true, in general, for higher order equations.

Corollary 3.2.38. *If $G_N[T] \geq 0$ on $I \times I$, then*

- $G_N[T](t, s) \leq 2 G_P[2T](2T - t, s)$ on $I \times I$.
- $0 \geq G_D[T](t, s) \geq -2 G_P[2T](2T - t, s)$ on $I \times I$.
- $G_N[T](t, s) \leq 2 G_N[2T](2T - t, s)$ on $I \times I$.
- $0 \geq G_{M_1}[T](t, s) \geq -2 G_N[2T](2T - t, s)$ on $I \times I$.

In particular, $G_P[2T](2T - t, s) \geq 0$ and $G_N[2T](2T - t, s) \geq 0$ on $I \times I$.

Proof. The inequalities are deduced from expressions (2.2.17) and (2.2.19) by taking into account that if $G_N[T] \geq 0$ on $I \times I$ then $G_D[T] \leq 0$ and $G_{M_1}[T] \leq 0$ on $I \times I$. \square

3.2.5. Global Order of Eigenvalues of Hill's Equation

It can also be proved that, when dealing with Hill's equation, there exists a certain order relation between the eigenvalues related to problems (N, T) , (D, T) , (M_1, T) and (M_2, T) .

Indeed, consider the following facts:

- (i) Let $\lambda_k^N[T], \lambda_{k+1}^N[T] \in \Lambda_N[T]$ be two consecutive eigenvalues of Neumann problem (N, T) and let $u_k^{N,T}$ and $u_{k+1}^{N,T}$ be their associated eigenfunctions. As we have seen in Section 3.1, the aforementioned eigenfunctions have k and $k + 1$ zeros on the interval $[0, T]$, respectively.

If we consider the even extensions of $u_k^{N,T}$ and $u_{k+1}^{N,T}$ to the interval $[0, 2T]$, it is clear that they have $2k$ and $2k + 2$ zeros on $[0, 2T]$, respectively, so there must exist an eigenvalue $\lambda \in \Lambda_N[2T]$, $\lambda_k^N[T] < \lambda < \lambda_{k+1}^N[T]$, such that its associated eigenfunction has exactly $2k + 1$ zeros on the interval $[0, 2T]$. From the decomposition of the Neumann spectrum showed in Section 2.3, we have that, necessarily, $\lambda \in \Lambda_{M_1}[T]$.

As we know that $\lambda_0^N[2T] = \lambda_0^N[T]$ we conclude that

$$\lambda_0^N[T] < \lambda_0^{M_1}[T] < \dots < \lambda_k^N[T] < \lambda_k^{M_1}[T] < \lambda_{k+1}^N[T] < \lambda_{k+1}^{M_1}[T] < \dots$$

- (ii) Analogously, we can easily see that $\Lambda_{M_2}[T]$ corresponds with eigenvalues of $\Lambda_D[2T]$ whose eigenfunctions have an even number of zeros on the interval

$(0, 2T)$ and $\Lambda_D[T]$ corresponds with eigenvalues of $\Lambda_D[2T]$ whose eigenfunctions have an odd number of zeros on $(0, 2T)$. Taking into account the fact that $\lambda_0^D[2T] = \lambda_0^{M_2}[T]$ we conclude that

$$\lambda_0^{M_2}[T] < \lambda_0^D[T] < \dots < \lambda_k^{M_2}[T] < \lambda_k^D[T] < \lambda_{k+1}^{M_2}[T] < \lambda_{k+1}^D[T] < \dots$$

(iii) Oscillation Theorem (Theorem 3.1.6) guarantees that the eigenvalues of periodic and antiperiodic problems related to the same interval always appear in the following order

$$\lambda_0^P[T] < \lambda_0^A[T] \leq \lambda_1^A[T] < \lambda_1^P[T] \leq \lambda_2^P[T] < \lambda_2^A[T] \leq \lambda_3^A[T] < \dots$$

Consequently, if we consider Item (iii) for problems $(P, 2T)$ and $(A, 2T)$ and we take into account the inequalities obtained in Items (i) and (ii) we can affirm that

- In each pair $\{\lambda_{2k-1}^P[2T], \lambda_{2k}^P[2T]\}$ of two consecutive eigenvalues of problem $(P, 2T)$, one of them belongs to $\Lambda_N[T]$ and the other one belongs to $\Lambda_D[T]$. In particular, if $\lambda_{2k-1}^P[2T] = \lambda_{2k}^P[2T]$ is a double eigenvalue, then it belongs to both $\Lambda_N[T]$ and $\Lambda_D[T]$.
- In each pair $\{\lambda_{2k}^A[2T], \lambda_{2k+1}^A[2T]\}$ of two consecutive eigenvalues of problem $(A, 2T)$, one of them belongs to $\Lambda_{M_1}[T]$ and the other one belongs to $\Lambda_{M_2}[T]$. As in the previous case, if $\lambda_{2k}^A[2T] = \lambda_{2k+1}^A[2T]$ is a double eigenvalue, then it belongs to both $\Lambda_{M_1}[T]$ and $\Lambda_{M_2}[T]$.

The previous reasoning lets us conclude that the eigenvalues of problem $(P, 4T)$ always appear in the following order:

$$\begin{aligned} \lambda_0^N[T] &< \{\lambda_0^{M_1}[T], \lambda_0^{M_2}[T]\} < \{\lambda_0^D[T], \lambda_1^N[T]\} \\ &< \{\lambda_1^{M_1}[T], \lambda_1^{M_2}[T]\} < \{\lambda_1^D[T], \lambda_2^N[T]\} < \dots \end{aligned}$$

As an immediate consequence we can also deduce an alternating relation between eigenvalues of (N, T) and (M_2, T) and also between those of (M_1, T) and (D, T) .

Corollary 3.2.39. *The following properties hold for any $a \in L^1(I)$.*

1. $\lambda_k^N[T] < \lambda_k^{M_2}[T] < \lambda_{k+1}^N[T] < \lambda_{k+1}^{M_2}[T], \quad k = 0, 1, \dots$
2. $\lambda_k^{M_1}[T] < \lambda_k^D[T] < \lambda_{k+1}^{M_1}[T] < \lambda_{k+1}^D[T], \quad k = 0, 1, \dots$

Remark 3.2.40. In [106, Chapter 1] the following equalities are proved in the case of an even potential on $[0, 2T]$:

$$u_1(2T, \lambda) = 2u_1(T, \lambda)u'_2(T, \lambda) - 1 = 1 + 2u'_1(T, \lambda)u_2(T, \lambda), \quad (3.2.8)$$

$$u'_1(2T, \lambda) = 2u_1(T, \lambda)u'_1(T, \lambda), \quad (3.2.9)$$

$$u_2(2T, \lambda) = 2u_2(T, \lambda)u'_2(T, \lambda), \quad (3.2.10)$$

$$u'_2(2T, \lambda) = u_1(2T, \lambda), \quad (3.2.11)$$

with u_1 and u_2 the fundamental solutions of Hill's equation defined in Theorem 3.1.6.

Moreover, it is easy to verify (see [23, Chapter 2] for the details) that

- $\lambda \in \Lambda_N[T]$ if and only if $u'_1(T, \lambda) = 0$.
- $\lambda \in \Lambda_D[T]$ if and only if $u_2(T, \lambda) = 0$.
- $\lambda \in \Lambda_{M_1}[T]$ if and only if $u_1(T, \lambda) = 0$.
- $\lambda \in \Lambda_{M_2}[T]$ if and only if $u'_2(T, \lambda) = 0$.

Therefore we deduce that, as \tilde{a} is an even function, the decomposition of Neumann and Dirichlet spectra in $2T$,

$$\Lambda_N[2T] = \Lambda_N[T] \cup \Lambda_{M_1}[T] \quad \text{and} \quad \Lambda_D[2T] = \Lambda_D[T] \cup \Lambda_{M_2}[T],$$

could also be deduced from the equalities (3.2.9) and (3.2.10). This deduction, despite being more direct than the one presented in this work, does not give any information about the order of eigenvalues.

We will see now some examples of the different situations that we could find. To calculate the eigenvalues we will use the characterization of the spectra given in Remark 3.2.40.

Example 3.2.41. If we consider the constant case $a(t) = 0$, it is known that (see [18])

$$\lambda_0^P[2T] = \lambda_0^N[T] = 0 \quad \text{and} \quad \lambda_0^A[2T] = \lambda_0^D[2T] = \left(\frac{\pi}{2T}\right)^2.$$

Moreover, denoting $\lambda = m^2 > 0$ and using [24] we obtain the explicit expressions of the corresponding Green's functions:

$$G_P[m^2, 2T](t, s) = \begin{cases} \frac{\cos(m(s - t + T))}{2m \sin mT}, & 0 \leq s \leq t \leq 2T, \\ \frac{\cos(m(s - t - T))}{2m \sin mT}, & 0 \leq t \leq s \leq 2T, \end{cases}$$

and

$$G_N[m^2, T](t, s) = \begin{cases} \frac{\cos(m s) \cos(m(T-t))}{m \sin m T}, & 0 \leq s \leq t \leq T, \\ \frac{\cos(m t) \cos(m(T-s))}{m \sin m T}, & 0 \leq t \leq s \leq T. \end{cases}$$

It is obvious that

$$G_N[m^2, T](0, 0) = 2 G_P[m^2, 2T](0, 0) = \frac{1}{m \tan m T}.$$

As a consequence, from Theorem 3.2.18, we know that $\lambda_0^{M_1}[T] = \left(\frac{\pi}{2T}\right)^2$. Moreover, from Theorem 3.2.18 and the fact that

$$G_N[m^2, T](T, T) = 2 G_P[m^2, 2T](T, T) = \frac{1}{m \tan m T},$$

we deduce that $\lambda_0^{M_2}[T] = \left(\frac{\pi}{2T}\right)^2$. This is also deduced from Corollary 2.3.1.

We can use [24] to calculate the Green's functions for the different boundary conditions

$$G_D[m^2, T](t, s) = \begin{cases} \frac{\sin(m s) \sin(m(t-T))}{m \sin m T}, & 0 \leq s \leq t \leq T, \\ \frac{\sin(m t) \sin(m(s-T))}{m \sin m T}, & 0 \leq t \leq s \leq T, \end{cases}$$

$$G_{M_1}[m^2, T](t, s) = \begin{cases} \frac{\cos(m s) \sin(m(t-T))}{m \cos m T}, & 0 \leq s \leq t \leq T, \\ \frac{\cos(m t) \sin(m(s-T))}{m \cos m T}, & 0 \leq t \leq s \leq T, \end{cases}$$

$$G_{M_2}[m^2, T](t, s) = \begin{cases} \frac{-\sin(m s) \cos(m(T-t))}{m \cos m T}, & 0 \leq s \leq t \leq T, \\ \frac{-\sin(m t) \cos(m(T-s))}{m \cos m T}, & 0 \leq t \leq s \leq T, \end{cases}$$

and

$$G_A[m^2, 2T](t, s) = \begin{cases} \frac{-\sin(m(s-t+T))}{2m \cos m T}, & 0 \leq s \leq t \leq T, \\ \frac{-\sin(m(-s+t+T))}{2m \cos m T}, & 0 \leq t \leq s \leq T. \end{cases}$$

We observe then that $\lambda_0^D[T] = \left(\frac{\pi}{T}\right)^2$.

In this case,

$$\Lambda_N[T] = \Lambda_D[T] \cup \{0\} = \Lambda_P[2T]$$

and

$$\Lambda_{M_1}[T] = \Lambda_{M_2}[T] = \Lambda_A[2T].$$

Then, if we represent graphically the discriminant (given in Oscillation Theorem (Theorem 3.1.6)), $\tilde{\Delta}(\lambda) = u_1(2T, \lambda) + u'_1(2T, \lambda)$, we obtain Figure 3.2.5.

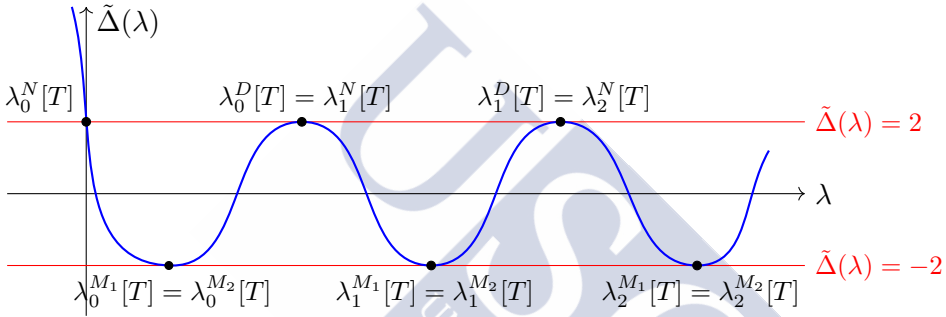


Figure 3.2.5: Graphic of $\tilde{\Delta}(\lambda)$ for $a(t) = 0$.

Example 3.2.42. If we consider $T = 2$ and

$$a(t) = \begin{cases} 0, & t \in [0, 1], \\ \frac{1}{10}, & t \in [1, 2], \end{cases}$$

the eigenvalues can be directly obtained and we can verify that

$$\lambda_0^N[2] = \lambda_0^N[4] = \lambda_0^P[4] \approx -0.0508,$$

$$\lambda_0^{M_2}[2] = \lambda_0^D[4] = \lambda_0^A[4] \approx 0.5346,$$

$$\lambda_0^{M_1}[2] \approx 0.5984$$

and

$$\lambda_0^D[2] \approx 2.4170.$$

Graphically, the situation would be represented in Figure 3.2.6.

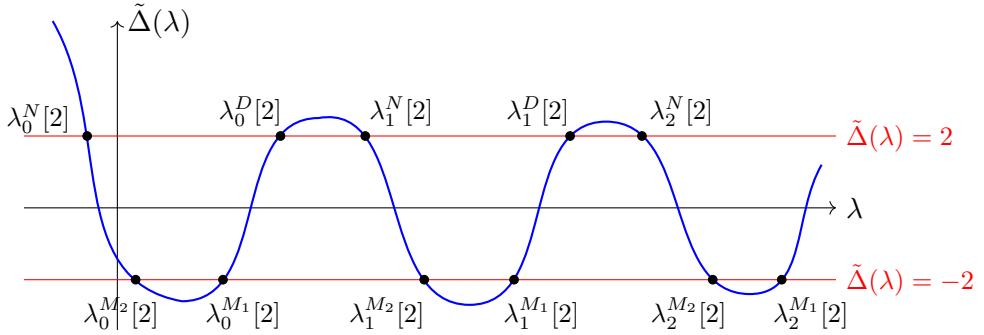


Figure 3.2.6: Graphic of $\tilde{\Delta}(\lambda)$ for a piecewise constant potential a .

Note that the k -th eigenvalue of problem (M_2, T) always appears before the one of problem (M_1, T) . In addition, the order between the eigenvalues of (N, T) and (D, T) is also maintained.

Example 3.2.43. Considering $T = \pi$ and $a(t) = \cos t$, we obtain the following approximations

$$\lambda_0^N[\pi] = \lambda_0^N[2\pi] = \lambda_0^D[2\pi] = \lambda_0^D[4\pi] \approx -0.378,$$

$$\lambda_0^{M_1}[\pi] = \lambda_0^A[2\pi] \approx -0.348,$$

$$\lambda_0^{M_2}[\pi] = \lambda_0^D[2\pi] \approx 0.5948$$

and

$$\lambda_0^D[\pi] \approx 0.918.$$

Graphically we would obtain Figure 3.2.7.

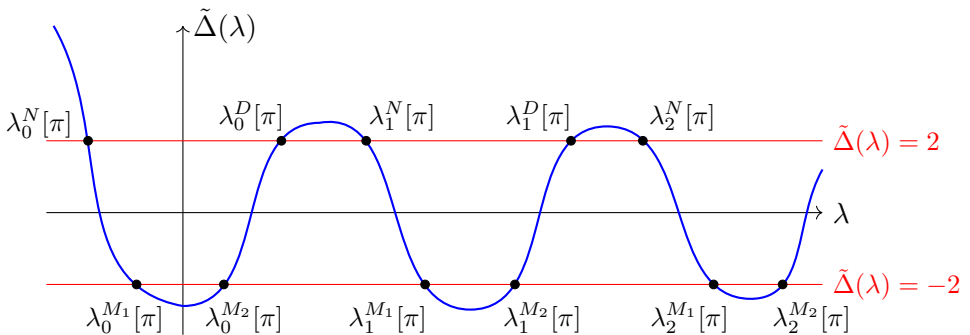


Figure 3.2.7: Graphic of $\tilde{\Delta}(\lambda)$ for $a(t) = \cos t$.

In this case, the k -th eigenvalue of (M_1, T) is smaller than the one of (M_2, T) . Again, the order between the eigenvalues of (N, T) and (D, T) is maintained.

The following example shows that eigenvalues related to problem (N, T) do not necessarily have to alternate with the ones of (D, T) .

Example 3.2.44. Considering $T = \pi$ and $a(t) = \cos 2t$, we obtain the following approximation for the spectra of the considered problems

$$\Lambda_P[4\pi] = \{-0.1218, 0.0923, 0.47065, 1.4668, 2.34076, 3.9792, 4.1009, \dots\},$$

$$\Lambda_P[2\pi] = \{-0.1218, 0.47065, 1.4668, 3.9792, 4.1009, \dots\},$$

$$\Lambda_{M_1}[\pi] = \Lambda_{M_2}[\pi] = \Lambda_A[2\pi] = \{0.0923, 2.34076, \dots\},$$

$$\Lambda_N[\pi] = \{-0.1218, 0.47065, 4.1009, \dots\}$$

and

$$\Lambda_D[\pi] = \{1.4668, 3.9792, \dots\}.$$

We observe that in this case

$$\lambda_0^N[\pi] < \lambda_1^N[\pi] < \lambda_0^D[\pi] < \lambda_1^D[\pi] < \lambda_2^N[\pi].$$

Note that the eigenvalues of mixed problems coincide. This is due to the fact that $a(t) = a(\pi - t)$ (see Corollary 2.3.1). Consequently, all the eigenvalues of $\Lambda_A[2\pi]$ are a double root of $\tilde{\Delta}(\lambda) = -2$.

Graphically we get Figure 3.2.8.

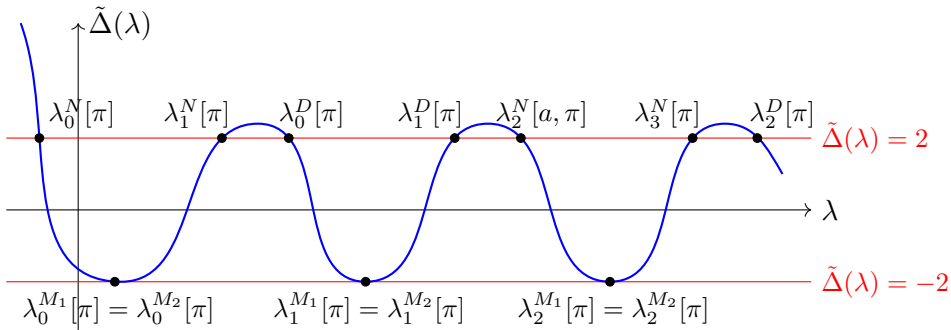


Figure 3.2.8: Graphic of $\tilde{\Delta}(\lambda)$ for $a(t) = \cos 2t$.

Remark 3.2.45. *The numerical results obtained in the considered examples suggest an order of eigenvalues even more precise than the one theoretically proved.*

It is observed that the eigenvalues of mixed problems alternate, with one eigenvalue of a mixed problem between two consecutive eigenvalues of the other one, and reciprocally. This has been observed in all the considered examples in which the spectra of the two mixed problems are different (Examples 3.2.42 and 3.2.43), independently of which of them appears first.

We also appreciate in the examples an alternation between Neumann and Dirichlet eigenvalues except for the case in which the spectrum of the mixed problems is the same (in this case the order of appearance of Dirichlet and Neumann changes between one pair of eigenvalues and the next one, as we can see in Example 3.2.44).

This situation suggests the existence of some property justifying this fact. However, up to this moment, this has not been formally proved and these speculations are uniquely based on the numerical results obtained while working with different potentials.

3.2.6. Explicit Criteria to Ensure Constant Sign of Green's Functions

As we have commented before, being able to ensure the constant sign of the Green's function is important as, among other things, in some cases it allows to warrant the constant sign of the solutions.

Moreover, as it has been mentioned at the beginning of this section, the periodic problem related to Hill's equation has been widely studied. In particular, many characterizations of maximum and antimaximum principles have been proved. All these criteria, by virtue of Theorem 1.1.7, can be used to ensure the constant sign of the Green's function related to problem (P, T) . We will compile next these criteria and, for the sake of simplicity, we will formulate them in terms of the constant sign of the Green's function, although most of them are originally proved for maximum and antimaximum principles.

First, we will introduce now some notation that we will use in this section:

- The positive part

$$h_+(t) = \max\{h(t), 0\}, \quad t \in I$$

and the negative one

$$h_-(t) = -\min\{h(t), 0\}, \quad t \in I,$$

are defined as usual.

- Given $1 \leq \alpha \leq \infty$ we will denote by α^* its conjugate, that is, the number satisfying the relation $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$. If $\alpha = 1$ then $\alpha^* = \infty$ and vice-versa.

- Finally, denoting by $H_0^1(I)$ the usual Sobolev space of the $W^{1,2}(I)$ functions that satisfy the Dirichlet conditions, we define $K(\alpha, T)$ as the best Sobolev constant in the inequality

$$C \|u\|_\alpha^2 \leq \|u'\|_2^2 \quad \text{for all } u \in H_0^1(I),$$

which is given explicitly by

$$K(\alpha, T) = \begin{cases} \frac{2\pi}{\alpha T^{1+\frac{2}{\alpha}}} \left(\frac{2}{2+\alpha} \right)^{1-\frac{2}{\alpha}} \left(\frac{\Gamma(\frac{1}{\alpha})}{\Gamma(\frac{1}{2}+\frac{1}{\alpha})} \right)^2, & 1 \leq \alpha < \infty, \\ \frac{4}{T}, & \alpha = \infty. \end{cases}$$

This expression was first given by Talenti in 1976 [138], but he did not prove it. The proof of the result can be seen in [51] for the case $1 < \alpha < \infty$ and in [26] for the cases $\alpha = 1$ and $\alpha = \infty$. Moreover, a detailed proof can also be found in [23, Appendix A].

We will compile now the aforementioned criteria.

Lemma 3.2.46. *Suppose that $a \in L^1(I)$, then:*

- (i) $\lambda_0^P[T] \leq -\frac{1}{T} \int_0^T a(s) \, ds$ and the equality holds if and only if a is constant. (See [160]).
- (ii) If $\|a_+\|_\alpha \leq K(2\alpha^*, T)$, then

$$\lambda_0^A[T] \geq \left(\frac{\pi}{T} \right)^2 \left(1 - \frac{\|a_+\|_\alpha}{K(2\alpha^*, T)} \right) \geq 0.$$

(See [162]).

Lemma 3.2.47 ([21, Proposition 3.1]). *If $G_P[T] < 0$ on $I \times I$ then $\int_0^T a(s) \, ds < 0$.*

Lemma 3.2.48 ([19, Theorem 3.1]). *If $\int_0^T a(t) \, dt \geq 0$, $a \not\equiv 0$ on I and*

$$\|a_+\|_{L^\alpha(I)} \leq K(2\alpha^*, T),$$

then $G_P[T] \geq 0$ on $I \times I$.

Lemma 3.2.49 ([142, Corollary 2.2]). *If $a \prec 0$, then $G_P[T] < 0$ on $I \times I$.*

Lemma 3.2.50 ([70, Corollary 2.5]). *If $a \in L^1(I)$, $a \not\equiv 0$ on I and*

$$\int_0^T a_+(s) \, ds < \frac{4}{T}, \quad \frac{\int_0^T a_+(s) \, ds}{1 - \frac{T}{4} \int_0^T a_+(s) \, ds} \leq \int_0^T a_-(s) \, ds,$$

then $G_P[T] < 0$ on $I \times I$.

We can also relate the discriminant $\Delta(\lambda)$, defined in Theorem 3.1.6, with the constant sign of the Green's function.

Lemma 3.2.51 ([21, Theorem 4.2]). *We have the following properties:*

1. $G_P[T] < 0$ on $I \times I$ if and only if $\Delta(\lambda) > 2$ for all $\lambda \leq 0$.
2. $G_P[T] \geq 0$ on $I \times I$ if and only if $\Delta(\lambda) > -2$ for all $\lambda < 0$ and $\Delta(0) < 2$.

Lemma 3.2.52 ([21, Theorem 4.3]). *Suppose that $\|\hat{a}_+\|_{L^\alpha(I)} \leq K(2\alpha^*, T)$, with $a(t) = \hat{a}(t) + \lambda$, where \hat{a} has mean zero and $\lambda = \frac{1}{T} \int_0^T a(s) \, ds$ is the mean value of a . Then*

1. $G_P[T] < 0$ on $I \times I$ if and only if $\int_0^T a(s) \, ds < 0$ and $\Delta(0) > 2$.
2. If $\int_0^T a(s) \, ds < 0$ and $\Delta(0) < 2$, then $G_P[T] \geq 0$ on $I \times I$.
3. If operator L coupled with periodic boundary conditions is nonresonant and

$$0 \leq \int_0^T a(s) \, ds \leq \frac{\pi^2}{T} \left(1 - \frac{\|\hat{a}_+\|_{L^\alpha(I)}}{K(2\alpha^*, T)} \right),$$

then $G_P[T] \geq 0$ on $I \times I$.

Now, as a corollary of Lemma 3.2.46, we obtain the following bounds for the first eigenvalues of Neumann, Dirichlet and mixed problems.

Corollary 3.2.53. *Let $a \in L^1(I)$. Then:*

1. $\lambda_0^N[T] \leq -\frac{1}{T} \int_0^T a(s) \, ds$ and the equality holds if and only if a is constant.
2. If $\|a_+\|_{L^\alpha(I)} \leq 2^{-1/\alpha} K(2\alpha^*, 2T)$ then

$$\lambda_0^D[T] > \min \left\{ \lambda_0^{M_1}[T], \lambda_0^{M_2}[T] \right\} \geq \left(\frac{\pi}{2T} \right)^2 \left(1 - \frac{2^{1/\alpha} \|a_+\|_{L^\alpha(I)}}{K(2\alpha^*, 2T)} \right) \geq 0.$$

Proof. 1. As a consequence of item (i) in Lemma 3.2.46 applied to the even extension of a we know that

$$\lambda_0^P[2T] \leq -\frac{1}{2T} \int_0^{2T} \tilde{a}(s) \, ds$$

and the equality holds only when \tilde{a} is constant. Assertion 1 in Theorem 3.2.18 warrants that $\lambda_0^N[T] = \lambda_0^P[2T]$. Then, we deduce that

$$\begin{aligned} \lambda_0^N[T] &\leq -\frac{1}{2T} \int_0^{2T} \tilde{a}(s) \, ds = -\frac{1}{2T} \left(\int_0^T a(s) \, ds + \int_T^{2T} a(2T-s) \, ds \right) \\ &= -\frac{1}{2T} \left(2 \int_0^T a(s) \, ds \right) = -\frac{1}{T} \int_0^T a(s) \, ds \end{aligned}$$

and the equality holds if and only if a is constant.

2. From Assertion (ii) in Lemma 3.2.46 we have that if $\|\tilde{a}_+\|_{L^\alpha(J)} \leq K(2\alpha^*, 2T)$ then

$$\lambda_0^A[2T] \geq \left(\frac{\pi}{2T}\right)^2 \left(1 - \frac{\|\tilde{a}_+\|_{L^\alpha(J)}}{K(2\alpha^*, 2T)}\right) \geq 0.$$

We also have that

$$\|\tilde{a}_+\|_{L^\alpha(J)}^\alpha = \int_0^{2T} |\tilde{a}_+(s)|^\alpha \, ds = 2 \int_0^T |a_+(s)|^\alpha \, ds = 2 \|a_+\|_{L^\alpha(I)}^\alpha,$$

from where we deduce that

$$\|\tilde{a}_+\|_{L^\alpha(J)} = 2^{1/\alpha} \|a_+\|_{L^\alpha(I)}.$$

The result can be concluded from the fact that

$$\lambda_0^A[2T] = \min \left\{ \lambda_0^{M_1}[T], \lambda_0^{M_2}[T] \right\} < \lambda_0^D[T],$$

proved in Theorem 2.3.2. □

On the other hand, using the implications between the constant sign of the Green's functions of the different problems, formulated in Corollary 3.2.22, we can rewrite Lemmas 3.2.47, 3.2.48, 3.2.49 and 3.2.50. For that, it is enough to consider those lemmas in terms of \tilde{a} and take into account the following relations (which are deduced from the previous proof):

- Condition $a \succ 0$ (respectively, $\prec 0$) is equivalent to $\tilde{a} \succ 0$ (respectively, $\prec 0$).
- We have the following relation between the norms of a and \tilde{a}

$$\|\tilde{a}\|_{L^\alpha(J)} = 2^{1/\alpha} \|a\|_{L^\alpha(I)}.$$

- As \tilde{a} is an even function, it verifies that

$$\int_0^{2T} \tilde{a}(s) \, ds = 2 \int_0^T a(s) \, ds.$$

We are now in conditions to rewrite the hypotheses in the corresponding terms for each case.

Corollary 3.2.54. *The following assertions hold:*

- (i) If $a \prec 0$, then $G_N[T] < 0$ on $I \times I$.
- (ii) If $a \in L^1(I)$, $\int_0^T a(t) \, dt \geq 0$, $a \not\equiv 0$ and $\|a_+\|_{L^\alpha(I)} \leq 2^{-1/\alpha} K(2\alpha^*, 2T)$, then $G_N[T] \geq 0$ on $I \times I$.
- (iii) If $a \in L^1(I)$, $a \not\equiv 0$ and

$$\int_0^T a_+(s) \, ds < \frac{1}{T}, \quad \frac{\int_0^T a_+(s) \, ds}{1 - T \int_0^T a_+(s) \, ds} \leq \int_0^T a_-(s) \, ds,$$

then $G_N[T] < 0$ on $I \times I$.

Any of the previous conditions implies that:

1. $G_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T) \times [0, T)$.
2. $G_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
3. $G_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
4. $G_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

Corollary 3.2.55. *If $G_N[T] < 0$ on $I \times I$ then $\int_0^T a(s) \, ds < 0$.*

Considering now the discriminant $\tilde{\Delta}(\lambda) = u_1(2T, \lambda) + u'_2(2T, \lambda)$ for the periodic problem on $[0, 2T]$ with potential \tilde{a} , from Lemmas 3.2.51 and 3.2.52, it is possible to obtain results about the constant sign of non-periodic Green's functions.

Corollary 3.2.56. *The following properties hold:*

- (i) $G_N[T] < 0$ on $I \times I$ if and only if $\tilde{\Delta}(\lambda) > 2$ for all $\lambda \leq 0$.
- (ii) $G_N[T] \geq 0$ on $I \times I$ if and only if $\tilde{\Delta}(\lambda) > -2$ for all $\lambda < 0$ and $\tilde{\Delta}(0) < 2$.

Corollary 3.2.57. *If $\tilde{\Delta}(\lambda) > -2$ for all $\lambda < 0$ then:*

- 1. $G_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T) \times [0, T)$.
- 2. $G_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
- 3. $G_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
- 4. $G_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

Corollary 3.2.58. *Let $a(t) = \hat{a}(t) + \lambda$, where \hat{a} has mean zero and $\lambda = \frac{1}{T} \int_0^T a(s) \, ds$ is the mean value of a , and suppose that $\|\hat{a}_+\|_{L^\alpha(I)} \leq 2^{-1/\alpha} K(2\alpha^*, 2T)$. Then:*

- (i) $G_N[T] < 0$ on $I \times I$ if and only if $\int_0^T a(s) \, ds < 0$ and $\tilde{\Delta}(0) > 2$.
- (ii) If $\int_0^T a(s) \, ds < 0$ and $\tilde{\Delta}(0) < 2$, then $G_N[T] \geq 0$ on $I \times I$.
- (iii) If operator L coupled with periodic boundary conditions on $[0, 2T]$ is nonresonant and

$$0 \leq \int_0^T a(s) \, ds \leq \frac{\pi^2}{4T} \left(1 - \frac{2^{1/\alpha} \|\hat{a}_+\|_{L^\alpha(I)}}{K(2\alpha^*, 2T)} \right),$$

then $G_N[T] \geq 0$ on $I \times I$.

Corollary 3.2.59. *If either*

$$\int_0^T a(s) \, ds < 0$$

or

$$0 \leq \int_0^T a(s) \, ds \leq \frac{\pi^2}{4T} \left(1 - \frac{2^{1/\alpha} \|\hat{a}_+\|_{L^\alpha(I)}}{K(2\alpha^*, 2T)} \right),$$

then:

- 1. $G_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T) \times [0, T)$.
- 2. $G_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
- 3. $G_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
- 4. $G_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

3.3. General Second Order Equation

In this section a more general problem of order 2 is considered and it is proved to be equivalent to the one involving Hill's equation treated in previous section. As a consequence, all the results obtained until the moment could be rewritten in terms of this problem.

First, the periodic problem will be studied and later the obtained results will be generalized to other boundary conditions.

All the results in this section can be found in [23, Section 3.5].

3.3.1. Periodic Problem

Consider now the general second order equation given in self-adjoint form

$$\begin{cases} (p u')'(t) + \bar{a}(t) u(t) = \bar{\sigma}(t), & \text{a. e. } t \in I, \\ u(0) = u(T), \quad (p u')(0) = (p u')(T), \end{cases} \quad (3.3.1)$$

with $p > 0$ a. e. $t \in I$ and $\frac{1}{p} \in L^1(I)$.

Let \bar{a} and $\bar{\sigma}$ be such that $\bar{a} p^{\frac{\alpha-1}{\alpha}}$ and $\bar{\sigma} p^{\frac{\alpha-1}{\alpha}} \in L^\alpha(I)$, for some $\alpha \in [1, \infty]$. Moreover, let u be a solution of problem (3.3.1), that is, a function $u \in \mathcal{AC}(I)$ such that $p u' \in \mathcal{AC}(I)$ and u satisfies the equalities in (3.3.1).

Defining

$$w(t) = \int_0^t \frac{ds}{p(s)}, \quad t \in I,$$

we have that $w \in \mathcal{AC}(I)$ and $w'(t) = \frac{1}{p(t)} > 0$ a. e. $t \in I$. Moreover, $w(0) = 0$ and $w(T) = R > 0$. So, from [37, Lemma 1], both w and w^{-1} are absolutely continuous.

As a consequence, both w and w^{-1} are continuous and strictly increasing functions on their intervals of definition.

Let $y : [0, R] \equiv K \rightarrow \mathbb{R}$ be defined as $y(t) = u(w^{-1}(t))$. Obviously, $y \in C(K)$ and, since $u \in \mathcal{AC}(I)$, $w^{-1} \in \mathcal{AC}(K)$ and $w^{-1} : K \rightarrow I$ is a monotone function, the following theorem warrants that $y = u \circ w^{-1} \in \mathcal{AC}(K)$.

Theorem 3.3.1. [113, Theorem 9.3] *Let $g \in \mathcal{AC}([c, d])$ be such that $g([c, d]) \subset [\alpha, \beta]$ and $f \in \mathcal{AC}([\alpha, \beta])$. If any of the two following conditions is satisfied*

(i) *g is monotone,*

(ii) *f is Lipschitz,*

then $f \circ g \in \mathcal{AC}([c, d])$.

Moreover, if $f \in \mathcal{AC}([\alpha, \beta])$ and $g \in \mathcal{AC}([c, d])$ is monotone, it is verified that

$$(f \circ g)'(t) = f'(g(t)) g'(t) \quad \text{a. e. } t \in [c, d].$$

This result can be seen in [37, Remark 3] and is deduced from [113, Theorems 9.3 and 38.4].

Therefore, for a. e. $t \in K$, the following equality is satisfied:

$$y'(t) = u'(w^{-1}(t)) (w^{-1}(t))' = u'(w^{-1}(t)) \frac{1}{w'(w^{-1}(t))} = (p u') (w^{-1}(t)).$$

In an analogous way it can be deduced that $(p u') \circ w^{-1} \in \mathcal{AC}(K)$ and

$$y''(t) = ((p u') \circ w^{-1})'(t) = (p u')' (w^{-1}(t)) p(w^{-1}(t)) \in L^1(I).$$

Consequently, we have that, for a. e. $t \in K$,

$$\begin{aligned} y''(t) + p(w^{-1}(t)) \bar{a}(w^{-1}(t)) y(t) &= p(p u')' (w^{-1}(t)) + (p \bar{a}) (w^{-1}(t)) u(w^{-1}(t)) \\ &= (p \bar{\sigma}) (w^{-1}(t)), \end{aligned}$$

and, moreover,

$$y(0) = u(w^{-1}(0)) = u(0) = u(T) = u(w^{-1}(R)) = y(R),$$

$$y'(0) = \lim_{t \rightarrow 0^+} y'(t) = \lim_{t \rightarrow 0^+} (p u') (w^{-1}(t)) = \lim_{s \rightarrow 0^+} (p u')(s) = (p u')(0)$$

and

$$y'(R) = \lim_{t \rightarrow R^-} y'(t) = \lim_{t \rightarrow R^-} (p u') (w^{-1}(t)) = \lim_{s \rightarrow T^-} (p u')(s) = (p u')(T).$$

On the other hand, note that

$$\begin{aligned} \int_0^R |p(w^{-1}(t)) \bar{a}(w^{-1}(t))|^\alpha dt &= \int_0^R \left| \bar{a} p^{\frac{\alpha-1}{\alpha}} (w^{-1}(t)) \right|^\alpha p(w^{-1}(t)) dt \\ &= \int_0^R \left| \bar{a} p^{\frac{\alpha-1}{\alpha}} (w^{-1}(t)) \right|^\alpha (w^{-1}(t))' dt \\ &= \int_{w^{-1}(0)}^{w^{-1}(R)} \left| \bar{a} p^{\frac{\alpha-1}{\alpha}} (s) \right|^\alpha ds = \int_0^T \left| \bar{a} p^{\frac{\alpha-1}{\alpha}} (s) \right|^\alpha ds \\ &= \left\| \bar{a} p^{\frac{\alpha-1}{\alpha}} \right\|_{L^\alpha(I)}^\alpha < +\infty, \end{aligned}$$

that is,

$$a(t) \equiv p(w^{-1}(t)) \bar{a}(w^{-1}(t)) \in L^\alpha(K).$$

Remark 3.3.2. *The fact that a is measurable is deduced from both $p \circ w^{-1}$ and $\bar{a} \circ w^{-1}$ being measurable. Indeed, we will see that if V is open, then*

$$(p \circ w^{-1})^{-1}(V) = w(p^{-1}(V))$$

is a measurable set.

As p is a measurable function, then $p^{-1}(V)$ is measurable. So, it is enough to verify that w takes measurable sets into measurable sets.

Since w is absolutely continuous, the following theorem guarantees that the image by w of any set of measure zero has measure zero.

Theorem 3.3.3 (Banach-Zarecki). [75, Theorem 18.25] *A function f is absolutely continuous on an interval $[c, d]$ if and only if the two following conditions are verified:*

- (i) *f is continuous and of bounded variation on $[c, d]$.*
- (ii) *The image through f of any subset of $[c, d]$ with measure zero is a set with measure zero.*

Moreover, as a consequence of [136, Chapter 6, Exercise 6] we have that, as w is a continuous function, the image of a set with measure zero has measure zero if and only if the image of any measurable set is measurable.

Consequently, $w(p^{-1}(V))$ is a measurable set and the function $p \circ w^{-1}$ is measurable.

An analogous reasoning could be considered for $\bar{a} \circ w^{-1}$.

Similarly we obtain that $\sigma(t) \equiv p(w^{-1}(t)) \bar{\sigma}(w^{-1}(t)) \in L^\alpha(K)$.

As a consequence, $y \in W^{2,1}(K)$ is a solution of

$$\begin{cases} y''(t) + a(t)y(t) = \sigma(t), & \text{a.e. } t \in K, \\ y(0) = y(R), \quad y'(0) = y'(R), \end{cases} \quad (3.3.2)$$

with $a, \sigma \in L^\alpha(K)$.

Reciprocally, let $a, \sigma \in L^\alpha(K)$ be arbitrary and let $y \in W^{2,1}(K)$ be a solution of problem (3.3.2). Consider functions p and w in the previous conditions.

Defining $u : I \rightarrow \mathbb{R}$ as $u(t) := y(w(t))$ and using the fact that $y \in \mathcal{AC}(K)$ and $w : I \rightarrow K$ is monotone and satisfies that $w \in \mathcal{AC}(I)$, we deduce from Theorem 3.3.1 that $u = y \circ w \in \mathcal{AC}(I)$.

Therefore, applying again the chain rule, we have that

$$u'(t) = y'(w(t)) w'(t) = y'(w(t)) \frac{1}{p(t)}, \quad \text{a.e. } t \in I,$$

that is,

$$(p u')(t) = y'(w(t)), \quad \text{a.e. } t \in I.$$

Since $y' \in \mathcal{AC}(K)$ and $w : I \rightarrow J$ is monotone and satisfies that $w \in \mathcal{AC}(I)$, we deduce again from Theorem 3.3.1 that $p u' = y' \circ w \in \mathcal{AC}(I)$, and

$$p(p u')'(t) = y''(w(t)), \quad \text{a.e. } t \in I.$$

Moreover

$$(p u')'(t) + \bar{a}(t) u(t) = \frac{y''(w(t))}{p(t)} + \bar{a}(t) y(w(t)) = \bar{\sigma}(t), \quad \text{a.e. } t \in I,$$

with $\bar{a}(t) = \frac{a(w(t))}{p(t)}$ and $\bar{\sigma}(t) = \frac{\sigma(w(t))}{p(t)}$, $t \in I$.

Obviously

$$u(0) = y(w(0)) = y(0) = y(R) = y(w(T)) = u(T). \quad (3.3.3)$$

The monotony assumptions on function w let us affirm that

$$p u'(0) = \lim_{t \rightarrow 0^+} (p u')(t) = \lim_{t \rightarrow 0^+} y'(w(t)) = \lim_{s \rightarrow 0^+} y'(s) = y'(0) \quad (3.3.4)$$

and

$$p u'(T) = \lim_{t \rightarrow T^-} (p u')(t) = \lim_{t \rightarrow T^-} y'(w(t)) = \lim_{s \rightarrow R^-} y'(s) = y'(R). \quad (3.3.5)$$

Finally, we observe that

$$\begin{aligned} \int_0^T \left| p^{\frac{\alpha-1}{\alpha}}(t) \bar{a}(t) \right|^\alpha dt &= \int_0^T \left| a(w(t)) p^{\frac{-1}{\alpha}}(t) \right|^\alpha dt \\ &= \int_0^T |a(w(t))|^\alpha \frac{dt}{p(t)} = \int_0^T |a(w(t))|^\alpha w'(t) dt \\ &= \int_{w(0)}^{w(T)} |a(s)|^\alpha ds = \int_0^R |a(s)|^\alpha ds \\ &= \|a\|_{L^\alpha(K)}^\alpha < +\infty. \end{aligned} \quad (3.3.6)$$

As a consequence $u \in \mathcal{AC}(I)$, with $p u' \in \mathcal{AC}(I)$, is a solution of problem (3.3.2) with $p^{\frac{\alpha-1}{\alpha}} \bar{a}$, $p^{\frac{\alpha-1}{\alpha}} \bar{\sigma} \in L^\alpha(I)$.

We have proved that problems (3.3.1) and (3.3.2) are equivalent and the qualitative properties of the solutions of both problems are the same.

We will see next the relation between the corresponding Green's functions.

Lemma 3.3.4. *Let $\bar{G}_P[T]$ and $G_P[R]$ be the Green's functions related to problems (3.3.1) and (3.3.2), respectively. It is verified that*

$$\bar{G}_P[T](t, s) = G_P[R](w(t), w(s)), \quad \forall (t, s) \in I \times I.$$

Proof. If u is the unique solution of problem (3.3.1), then, as we have proved before, $y(t) = u(w^{-1}(t))$ is the unique solution of (3.3.2) for $\sigma(t) = p(w^{-1}(t)) \bar{\sigma}(w^{-1}(t))$ and satisfies that

$$y(t) = \int_0^R G_P[R](t, s) p(w^{-1}(s)) \bar{\sigma}(w^{-1}(s)) \, ds.$$

Conversely, if y is the unique solution of (3.3.2) then, using previous arguments again, $u = y(w(t))$ is the unique solution of problem (3.3.1). Consequently

$$\begin{aligned} u(t) = y(w(t)) &= \int_0^R G_P[R](w(t), s) p(w^{-1}(s)) \bar{\sigma}(w^{-1}(s)) \, ds \\ &= \int_0^T G_P[R](w(t), w(s)) \bar{\sigma}(s) \, ds, \end{aligned}$$

from where we deduce the result. □

As an immediate consequence we obtain the following corollary.

Corollary 3.3.5. *The following equivalences hold:*

1. $\bar{G}_P[T] \geq 0$ on $I \times I$ if and only if $G_P[R] \geq 0$ on $K \times K$.
2. $\bar{G}_P[T] < 0$ on $I \times I$ if and only if $G_P[R] < 0$ on $K \times K$.

The previous result lets us obtain some criteria about the constant sign of the Green's function of problem (3.3.1) from Lemmas 3.2.47, 3.2.48, 3.2.49 and 3.2.50. These results are deduced by simply taking into account the following facts:

- As a consequence from (3.3.6) we have that

$$\|a\|_{L^\alpha(K)} = \left\| p^{\frac{\alpha-1}{\alpha}} \bar{a} \right\|_{L^\alpha(I)}.$$

- It is verified that

$$\begin{aligned} \int_0^R a(t) \, dt &= \int_0^R p(w^{-1}(t)) \bar{a}(w^{-1}(t)) \, dt = \int_0^R \bar{a}(w^{-1}(t)) (w^{-1}(t))' \, dt \\ &= \int_0^T \bar{a}(s) \, ds. \end{aligned}$$

- Since by hypothesis $p > 0$ a.e. $t \in I$, condition $a \succ 0$ (respectively, $\prec 0$) is equivalent to $\bar{a} \succ 0$ (respectively, $\prec 0$). Moreover,

$$a_+(t) = p(w^{-1}(t)) \bar{a}_+(w^{-1}(t)), \quad a_-(t) = p(w^{-1}(t)) \bar{a}_-(w^{-1}(t))$$

and consequently

$$\int_0^R a_+(t) \, dt = \int_0^T \bar{a}_+(t) \, dt \quad \text{and} \quad \int_0^R a_-(t) \, dt = \int_0^T \bar{a}_-(t) \, dt.$$

Corollary 3.3.6. *If $\bar{a} \prec 0$, then $\bar{G}_P[T] < 0$ on $I \times I$.*

Corollary 3.3.7. *If $\bar{a} \succ 0$ and $\left\| p^{\frac{\alpha-1}{\alpha}} \bar{a} \right\|_{L^\alpha(I)} \leq K(2\alpha^*, R)$, then $\bar{G}_P[T] \geq 0$ on $I \times I$.*

Corollary 3.3.8. *If $\int_0^T \bar{a}(t) \, dt \geq 0$, $\bar{a} \not\equiv 0$ on I and $\left\| p^{\frac{\alpha-1}{\alpha}} \bar{a}_+ \right\|_{L^\alpha(I)} \leq K(2\alpha^*, R)$, then $\bar{G}_P[T] \geq 0$ on $I \times I$.*

Corollary 3.3.9. *If $\bar{a} \in L^1(I)$, $\bar{a} \not\equiv 0$ on I and*

$$\int_0^T \bar{a}_+(s) \, ds < \frac{4}{R}, \quad \frac{\int_0^T \bar{a}_+(s) \, ds}{1 - \frac{R}{4} \int_0^T \bar{a}_+(s) \, ds} \leq \int_0^T \bar{a}_-(s) \, ds,$$

then $\bar{G}_P[T] < 0$ on $I \times I$.

Corollary 3.3.10. *If $\bar{G}_P[T](t, s) < 0$ for all $(t, s) \in I \times I$ then $\int_0^T \bar{a}(s) \, ds < 0$.*

Considering now the discriminant $\Delta(\lambda) = u_1(R, \lambda) + u_2'(R, \lambda)$ for problem (3.3.2), Lemmas 3.2.51 and 3.2.52 can be rewritten in order to obtain some conditions that assure the constant sign of the Green's function of problem (3.3.1).

Corollary 3.3.11. *The following properties hold:*

- (i) $\bar{G}_P[T] < 0$ on $I \times I$ if and only if $\Delta(\lambda) > 2$ for all $\lambda \leq 0$.
- (ii) $\bar{G}_P[T] \geq 0$ on $I \times I$ if and only if $\Delta(\lambda) > -2$ for all $\lambda < 0$ and $\Delta(0) < 2$.

Corollary 3.3.12. *Let $\bar{a}(t) = \hat{a}(t) + \lambda \frac{1}{p(t)}$, where $\lambda = \frac{1}{R} \int_0^T \bar{a}(s) \, ds$ is $\frac{T}{R}$ times the mean value of \bar{a} and suppose that*

$$\left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)} \leq K(2\alpha^*, R).$$

Then

- (i) $\bar{G}_P[T] < 0$ on $I \times I$ if and only if $\int_0^T \bar{a}(s) \, ds < 0$ and $\Delta(0) > 2$.
- (ii) If $\int_0^T \bar{a}(s) \, ds < 0$ and $\Delta(0) < 2$, then $\bar{G}_P[T] \geq 0$ on $I \times I$.
- (iii) If operator L under periodic boundary conditions is nonresonant and

$$0 \leq \int_0^T \bar{a}(s) \, ds \leq \frac{\pi^2}{R} \left(1 - \frac{\left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)}}{K(2\alpha^*, R)} \right),$$

then $\bar{G}_P[T] \geq 0$ on $I \times I$.

Proof. The hypotheses of Lemma 3.2.52 (applied to $a(t) = p(w^{-1}(t)) \bar{a}(w^{-1}(t))$) will be rewritten in terms of problem (3.3.1). Indeed, such result considers that $\|\hat{a}_+\|_{L^\alpha(K)} \leq K(2\alpha^*, R)$, with $a(t) = \hat{a}(t) + \lambda$, where $\lambda = \frac{1}{R} \int_0^R a(s) \, ds$ is the mean value of a . It is immediate to verify that

$$\lambda = \frac{1}{R} \int_0^T \bar{a}(s) \, ds$$

and, clearly, this is $\frac{T}{R}$ times the mean value of \bar{a} .

Moreover,

$$\begin{aligned} \|\hat{a}_+\|_{L^\alpha(K)}^\alpha &= \|(a - \lambda)_+\|_{L^\alpha(K)}^\alpha = \int_0^R (a(t) - \lambda)_+^\alpha \, dt \\ &= \int_0^R (p(w^{-1}(t)) \bar{a}(w^{-1}(t)) - \lambda)_+^\alpha \, dt \\ &= \int_0^T (p(s) \bar{a}(s) - \lambda)_+^\alpha \frac{1}{p(s)} \, ds \\ &= \int_0^T \left(p^{\frac{\alpha-1}{\alpha}} \left(\bar{a}(s) - \frac{\lambda}{p(s)} \right)_+ \right)^\alpha \, ds \\ &= \left\| p^{\frac{\alpha-1}{\alpha}} \left(\bar{a} - \frac{\lambda}{p} \right)_+ \right\|_{L^\alpha(I)}^\alpha = \left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)}^\alpha, \end{aligned}$$

that is,

$$\|\hat{a}_+\|_{L^\alpha(K)} = \left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)}.$$

The other changes in this corollary with respect to Lemma 3.2.52 are immediately obtained from the same considerations as in previous results. \square

3.3.2. Non-periodic Conditions

All the previous reasonings have been done considering periodic boundary conditions. Nevertheless, equalities (3.3.3), (3.3.4) and (3.3.5) guarantee that

$$u(0) = y(0), \quad u(T) = y(R),$$

and

$$(p u')(0) = y'(0), \quad (p u')(T) = y'(R).$$

Consequently, periodic conditions in problems (3.3.1) and (3.3.2) can be substituted by any other kind of boundary conditions and this does not affect to the equivalence of the problems. We obtain the same relation between the Green's functions corresponding to each case, that is, using an analogous notation to the periodic case, we have the following equalities:

$$\bar{G}_N[T](t, s) = G_N[R](w(t), w(s)), \quad \forall (t, s) \in I \times I,$$

$$\bar{G}_D[T](t, s) = G_D[R](w(t), w(s)), \quad \forall (t, s) \in I \times I,$$

$$\bar{G}_{M_1}[T](t, s) = G_{M_1}[R](w(t), w(s)), \quad \forall (t, s) \in I \times I,$$

$$\bar{G}_{M_2}[T](t, s) = G_{M_2}[R](w(t), w(s)) \quad \forall (t, s) \in I \times I$$

and

$$\bar{G}_A[T](t, s) = G_A[R](w(t), w(s)), \quad \forall (t, s) \in I \times I.$$

As a consequence all the results relating different Green's functions which were obtained for Hill's equation are still valid in this more general case. In particular, all the corollaries in Subsection 3.3.1 can be rewritten in terms of the Green's function of other boundary conditions, in an analogous way to what we have made in Subsection 3.2.6. To do that it is enough to consider Lemmas 3.2.47, 3.2.48, 3.2.49 and 3.2.50 for \tilde{a} (the even extension of a) and take into account the following considerations:

- We have that

$$\|\tilde{a}_+\|_{L^\alpha[0, 2R]} = 2^{1/\alpha} \|a_+\|_{L^\alpha(K)} = 2^{1/\alpha} \left\| p^{\frac{\alpha-1}{\alpha}} \bar{a}_+ \right\|_{L^\alpha(I)}.$$

- Condition $\tilde{a} \succ 0$ (respectively, $\prec 0$) is equivalent to $a \succ 0$ (respectively, $\prec 0$) which, at the same time, is equivalent to $\bar{a} \succ 0$ (respectively, $\prec 0$).

- The integrals of potentials present the following relation

$$\int_0^{2R} \tilde{a}(t) \, dt = 2 \int_0^R a(t) \, dt = 2 \int_0^T \bar{a}(t) \, dt.$$

Analogously, since $p > 0$ a.e. $t \in I$,

$$\int_0^{2R} \tilde{a}_+(t) \, dt = 2 \int_0^T \bar{a}_+(t) \, dt \quad \text{and} \quad \int_0^{2R} \tilde{a}_-(t) \, dt = 2 \int_0^T \bar{a}_-(t) \, dt.$$

Corollary 3.3.13. (i) If $\bar{a} \prec 0$, then $\bar{G}_N[T] < 0$ on $I \times I$.

(ii) If $\int_0^T \bar{a}(t) \, dt \geq 0$, $\bar{a} \not\equiv 0$ and $\left\| p^{\frac{\alpha-1}{\alpha}} \bar{a}_+ \right\|_{L^\alpha(I)} \leq 2^{-1/\alpha} K(2\alpha^*, 2R)$, then $\bar{G}_N[T] \geq 0$ on $I \times I$.

(iii) If $\bar{a} \in L^1(I)$, $\bar{a} \not\equiv 0$ and

$$\int_0^T \bar{a}_+(s) \, ds < \frac{1}{R}, \quad \frac{\int_0^T \bar{a}_+(s) \, ds}{1 - R \int_0^T \bar{a}_+(s) \, ds} \leq \int_0^T \bar{a}_-(s) \, ds,$$

then $\bar{G}_N[T] < 0$ on $I \times I$.

Any of the previous conditions implies that:

1. $\bar{G}_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T] \times [0, T]$.
2. $\bar{G}_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
3. $\bar{G}_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
4. $\bar{G}_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

Corollary 3.3.14. If $\bar{G}_N[T](t, s) < 0$ for all $(t, s) \in I \times I$ then $\int_0^T \bar{a}(s) \, ds < 0$.

Moreover, considering the discriminant $\tilde{\Delta}(\lambda) = u_1(2R, \lambda) + u'_2(2R, \lambda)$ for the periodic problem

$$\begin{cases} y''(t) + \tilde{a}(t)y(t) = \tilde{\sigma}(t), & \text{a.e. } t \in [0, 2R], \\ y(0) = y(2R), \quad y'(0) = y'(2R), \end{cases}$$

obtained from (3.3.2) by simply considering \tilde{a} and $\tilde{\sigma}$ the even extensions of a and σ , we can deduce results for Green's function different from the periodic one by rewriting Lemmas 3.2.51 and 3.2.52. The results are the following ones.

Corollary 3.3.15. *The following properties hold:*

- (i) $\bar{G}_N[T] < 0$ on $I \times I$ if and only if $\tilde{\Delta}(\lambda) > 2$ for all $\lambda \leq 0$.
- (ii) $\bar{G}_N[T] \geq 0$ on $I \times I$ if and only if $\tilde{\Delta}(\lambda) > -2$ for all $\lambda < 0$ and $\tilde{\Delta}(0) < 2$.

Corollary 3.3.16. *If $\tilde{\Delta}(\lambda) > -2$ for all $\lambda < 0$ then:*

1. $\bar{G}_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T) \times [0, T)$.
2. $\bar{G}_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
3. $\bar{G}_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
4. $\bar{G}_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

Corollary 3.3.17. *Suppose that*

$$\left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)} \leq 2^{-1/\alpha} K(2\alpha^*, 2R),$$

with $\bar{a}(t) = \hat{a}(t) + \lambda \frac{1}{p(t)}$, where $\lambda = \frac{1}{R} \int_0^T \bar{a}(s) \, ds$ is $\frac{T}{R}$ times the mean value of \bar{a} . Then

- (i) $\bar{G}_N[T] < 0$ on $I \times I$ if and only if $\int_0^T \bar{a}(s) \, ds < 0$ and $\tilde{\Delta}(0) > 2$.
- (ii) If $\int_0^T \bar{a}(s) \, ds < 0$ and $\tilde{\Delta}(0) < 2$, then $\bar{G}_N[T] \geq 0$ on $I \times I$.
- (iii) If operator L under periodic boundary conditions on $[0, 2T]$ is nonresonant and

$$0 \leq \int_0^T \bar{a}(s) \, ds \leq \frac{\pi^2}{4R} \left(1 - \frac{2^{1/\alpha} \left\| p^{\frac{\alpha-1}{\alpha}} \hat{a}_+ \right\|_{L^\alpha(I)}}{K(2\alpha^*, 2R)} \right),$$

then $\bar{G}_N[T] \geq 0$ on $I \times I$.

Proof. This result is obtained by applying Lemma 3.2.52 to \tilde{a} , the even extension of $a(t) = p(w^{-1}(t)) \bar{a}(w^{-1}(t))$. The hypotheses of that lemma consider that

$$\left\| \hat{a}_+ \right\|_{L^\alpha[0, 2R]} \leq K(2\alpha^*, 2R),$$

with $\tilde{a}(t) = \hat{a}(t) + \lambda$, where $\lambda = \frac{1}{2R} \int_0^{2R} \tilde{a}(s) \, ds$ is the mean value of \tilde{a} . From the fact that \tilde{a} is an even function and from the relation between a and \bar{a} we have that

$$\lambda = \frac{1}{R} \int_0^R a(s) \, ds = \frac{1}{R} \int_0^T \bar{a}(s) \, ds,$$

so λ is $\frac{T}{R}$ times the mean value of \bar{a} .

In addition, using again that \tilde{a} is even (and so $\hat{\tilde{a}}$ is even too) and taking into account the reasoning developed in the proof of Corollary 3.3.12, we arrive at the following relation between the norms of $\hat{\tilde{a}}$ and $\hat{\hat{a}}$

$$\left\| \hat{\tilde{a}}_+ \right\|_{L^\alpha[0,2R]} = 2^{1/\alpha} \left\| \hat{a}_+ \right\|_{L^\alpha(K)} = 2^{1/\alpha} \left\| p^{\frac{\alpha-1}{\alpha}} \hat{\hat{a}}_+ \right\|_{L^\alpha(I)}.$$

The rest of variations in this corollary with respect to Lemma 3.2.52 are immediately deduced from the considerations used in previous results. \square

Corollary 3.3.18. *If either*

$$\int_0^T \bar{a}(s) \, ds < 0$$

or

$$0 \leq \int_0^T \bar{a}(s) \, ds \leq \frac{\pi^2}{4R} \left(1 - \frac{2^{1/\alpha} \left\| p^{\frac{\alpha-1}{\alpha}} \hat{\hat{a}}_+ \right\|_{L^\alpha(I)}}{K(2\alpha^*, 2R)} \right),$$

then:

1. $\bar{G}_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, T) \times [0, T)$.
2. $\bar{G}_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, T] \times (0, T]$.
3. $\bar{G}_D[T](t, s) < 0$ for all $(t, s) \in (0, T) \times (0, T)$.
4. $\bar{G}_D[2T](t, s) < 0$ for all $(t, s) \in (0, 2T) \times (0, 2T)$.

We will finish this subsection with an example in which we will use the relation between the Green's functions of problems (3.3.1) and (3.3.2) to obtain the explicit expression of one of them through another one of a problem with constant coefficients.

Example 3.3.19. *Consider the equation*

$$\left(\frac{1}{t} u'(t) \right)' + \lambda t u(t) = 0, \quad t \in [0, 1], \quad (3.3.7)$$

$$u(0) = u(1), \quad \lim_{t \rightarrow 0^+} (t u'(t))(t) = \lim_{t \rightarrow 1^-} (t u'(t))(t), \quad (3.3.8)$$

which is a periodic problem of the type of (3.3.1) with $\bar{a}(t) = \lambda t$, $p(t) = \frac{1}{t}$ and $[0, T] = [0, 1]$.

With the definitions given at the beginning of this section, we have that

$$w(t) = \int_0^t s \, ds = \frac{t^2}{2}, \quad t \in [0, 1]$$

and

$$w^{-1}(t) = \sqrt{2t}, \quad t \in \left[0, \frac{1}{2}\right] \equiv [0, R],$$

so the previous problem is equivalent to the constant's coefficients periodic problem

$$\begin{cases} y''(t) + \lambda y(t) = \sigma(t), & \text{a.e. } t \in \left[0, \frac{1}{2}\right], \\ y(0) = y\left(\frac{1}{2}\right), \quad y'(0) = y'\left(\frac{1}{2}\right). \end{cases} \quad (3.3.9)$$

Using [24] we can calculate the Green's function related to problem (3.3.9). We will distinguish between the case $\lambda = m^2 > 0$ and $\lambda = -m^2 < 0$ ($\lambda = 0$ is not considered since it is an eigenvalue for this problem). We obtain

$$G_P[m^2, R](t, s) = \begin{cases} \frac{\cos\left(m\left(s - t + \frac{1}{4}\right)\right)}{2m \sin\left(\frac{m}{4}\right)}, & 0 \leq s \leq t \leq \frac{1}{2}, \\ \frac{\cos\left(m\left(s - t - \frac{1}{4}\right)\right)}{2m \sin\left(\frac{m}{4}\right)}, & 0 \leq t < s \leq \frac{1}{2}, \end{cases}$$

and

$$G_P[-m^2, R](t, s) = \begin{cases} \frac{\cosh\left(m\left(s - t + \frac{1}{4}\right)\right)}{2m \sinh\left(\frac{m}{4}\right)}, & 0 \leq s \leq t \leq \frac{1}{2}, \\ \frac{\cosh\left(m\left(s - t - \frac{1}{4}\right)\right)}{2m \sinh\left(\frac{m}{4}\right)}, & 0 \leq t < s \leq \frac{1}{2}. \end{cases}$$

Lemma 3.3.4 allows us to calculate the exact expression of the Green's function related to the periodic problem (3.3.7), (3.3.8). Such function is

$$\bar{G}_P[T](t, s) = \begin{cases} \frac{\cos\left(\frac{m}{2}\left(s^2 - t^2 + \frac{1}{2}\right)\right)}{2m \sin\left(\frac{m}{4}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cos\left(\frac{m}{2}\left(s^2 - t^2 - \frac{1}{2}\right)\right)}{2m \sin\left(\frac{m}{4}\right)}, & 0 \leq t < s \leq 1, \end{cases}$$

for $\lambda = m^2 > 0$ and

$$\bar{G}_P[T](t, s) = \begin{cases} \frac{\cosh\left(\frac{m}{2}\left(s^2 - t^2 + \frac{1}{2}\right)\right)}{2m \sinh\left(\frac{m}{4}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cosh\left(\frac{m}{2}\left(s^2 - t^2 - \frac{1}{2}\right)\right)}{2m \sinh\left(\frac{m}{4}\right)}, & 0 \leq t < s \leq 1, \end{cases}$$

for $\lambda = -m^2 < 0$.

Taking into account the eigenvalues calculated in Example 3.2.41 for a constant potential a , we can analyse the values of λ for which $G_P[\lambda, R]$ has constant sign. Lemma 3.3.4 warrants that

- $\bar{G}_P[T](t, s) < 0$ for all $(t, s) \in [0, 1] \times [0, 1]$ if and only if $\lambda < 0$.
- $\bar{G}_P[T](t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$ if and only if $0 < \lambda \leq (2\pi)^2$.

Analogously, the expressions obtained in Example 3.2.41 for $\lambda = m^2$ (and also the ones corresponding to $\lambda = 0$ or $\lambda = -m^2$ calculated with [24]) allow us to deduce the exact expression of the Green's functions related to equation (3.3.7) under different boundary conditions.

For Neumann problem

$$\begin{cases} \left(\frac{1}{t} u'(t) \right)' + \lambda t u(t) = 0, & t \in [0, 1], \\ \lim_{t \rightarrow 0^+} (t u'(t))(t) = \lim_{t \rightarrow 1^-} (t u'(t))(t) = 0, \end{cases}$$

we have that

$$\bar{G}_N[m^2 t, T](t, s) = \begin{cases} \frac{\cos\left(m \frac{s^2}{2}\right) \cos\left(\frac{m}{2}(1-t^2)\right)}{m \sin\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cos\left(m \frac{t^2}{2}\right) \cos\left(\frac{m}{2}(1-s^2)\right)}{m \sin\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$\bar{G}_N[-m^2 t, T](t, s) = \begin{cases} \frac{\cosh\left(m \frac{s^2}{2}\right) \cosh\left(\frac{m}{2}(1-t^2)\right)}{m \sinh\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cosh\left(m \frac{t^2}{2}\right) \cosh\left(\frac{m}{2}(1-s^2)\right)}{m \sinh\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and we deduce that

- $\bar{G}_N[T](t, s) < 0$ for all $(t, s) \in [0, 1] \times [0, 1]$ if and only if $\lambda < 0$.
- $\bar{G}_N[T](t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$ if and only if $0 < \lambda \leq \pi^2$.

The Green's function related to Dirichlet problem

$$\begin{cases} \left(\frac{1}{t} u'(t) \right)' + \lambda t u(t) = 0, & t \in [0, 1], \\ u(0) = u(1) = 0, \end{cases}$$

is

$$\begin{aligned} \bar{G}_D[m^2 t, T](t, s) &= \begin{cases} \frac{\sin\left(m \frac{s^2}{2}\right) \sin\left(\frac{m}{2}(t^2 - 1)\right)}{m \sin\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\sin\left(m \frac{t^2}{2}\right) \sin\left(\frac{m}{2}(s^2 - 1)\right)}{m \sin\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ \bar{G}_D[0, T](t, s) &= \frac{1}{2} \begin{cases} s^2 (t^2 - 1), & 0 \leq s \leq t \leq 1, \\ t^2 (s^2 - 1), & 0 \leq t \leq s \leq 1, \end{cases} \\ \bar{G}_D[-m^2 t, T](t, s) &= \begin{cases} \frac{\sinh\left(m \frac{s^2}{2}\right) \sinh\left(\frac{m}{2}(t^2 - 1)\right)}{m \sinh\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\sinh\left(m \frac{t^2}{2}\right) \sinh\left(\frac{m}{2}(s^2 - 1)\right)}{m \sinh\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases} \end{aligned}$$

and we have that

- $G_D[T](t, s) < 0$ for all $(t, s) \in (0, 1) \times (0, 1)$ if and only if $\lambda < (2\pi)^2$.

With regard to mixed problems M_1 ,

$$\begin{cases} \left(\frac{1}{t} u'(t) \right)' + \lambda t u(t) = 0, & t \in [0, 1], \\ \lim_{t \rightarrow 0^+} (t u'(t))(t) = u(1) = 0, \end{cases}$$

and M_2 ,

$$\begin{cases} \left(\frac{1}{t} u'(t) \right)' + \lambda t u(t) = 0, & t \in [0, 1], \\ u(0) = \lim_{t \rightarrow 1^-} (t u'(t))(t) = 0, \end{cases}$$

the corresponding Green's functions are

$$\bar{G}_{M_1}[m^2 t, T](t, s) = \begin{cases} \frac{\cos\left(m \frac{s^2}{2}\right) \sin\left(\frac{m}{2}(t^2 - 1)\right)}{m \cos\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cos\left(m \frac{t^2}{2}\right) \sin\left(\frac{m}{2}(s^2 - 1)\right)}{m \cos\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$\bar{G}_{M_1}[0, T](t, s) = \begin{cases} \frac{t^2 - 1}{2}, & 0 \leq s \leq t \leq 1, \\ \frac{s^2 - 1}{2}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$\bar{G}_{M_1}[-m^2 t, T](t, s) = \begin{cases} \frac{\cosh\left(m \frac{s^2}{2}\right) \sinh\left(\frac{m}{2}(t^2 - 1)\right)}{m \cosh\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{\cosh\left(m \frac{t^2}{2}\right) \sinh\left(\frac{m}{2}(s^2 - 1)\right)}{m \cosh\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$\bar{G}_{M_2}[m^2 t, T](t, s) = \begin{cases} \frac{-\sin\left(m \frac{s^2}{2}\right) \cos\left(\frac{m}{2}(1 - t^2)\right)}{m \cos\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{-\sin\left(m \frac{t^2}{2}\right) \cos\left(\frac{m}{2}(1 - s^2)\right)}{m \cos\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$\bar{G}_{M_2}[0, T](t, s) = \begin{cases} \frac{-s^2}{2}, & 0 \leq s \leq t \leq 1, \\ \frac{-t^2}{2}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$\bar{G}_{M_2}[-m^2 t, T](t, s) = \begin{cases} \frac{-\sinh\left(m \frac{s^2}{2}\right) \cosh\left(\frac{m}{2}(1 - t^2)\right)}{m \cosh\left(\frac{m}{2}\right)}, & 0 \leq s \leq t \leq 1, \\ \frac{-\sinh\left(m \frac{t^2}{2}\right) \cosh\left(\frac{m}{2}(1 - s^2)\right)}{m \cosh\left(\frac{m}{2}\right)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

In this case we have that

- $G_{M_1}[T](t, s) < 0$ for all $(t, s) \in [0, 1) \times [0, 1)$ if and only if $\lambda < \pi^2$.
- $G_{M_2}[T](t, s) < 0$ for all $(t, s) \in (0, 1] \times (0, 1]$ if and only if $\lambda < \pi^2$.

In addition, the expression of each Green's function lets us calculate the spectrum of the problem under the corresponding boundary conditions. This way we obtain that

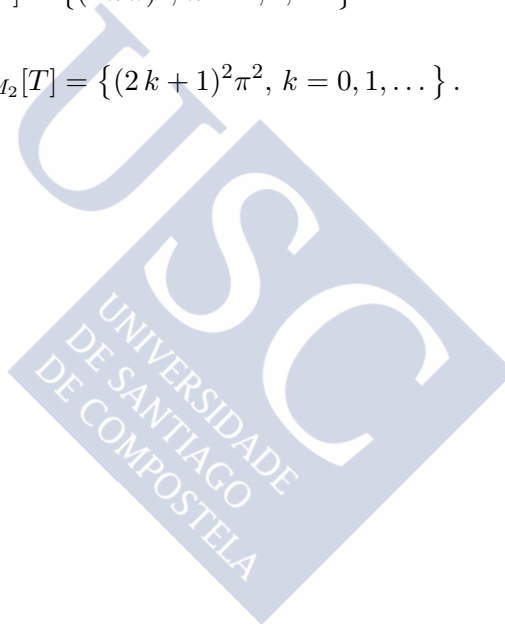
$$\Lambda_P[T] = \{(4k\pi)^2, k = 0, 1, \dots\},$$

$$\Lambda_N[T] = \{(2k\pi)^2, k = 0, 1, \dots\},$$

$$\Lambda_D[T] = \{(2k\pi)^2, k = 1, 2, \dots\}$$

and

$$\Lambda_{M_1}[T] = \Lambda_{M_2}[T] = \{(2k+1)^2\pi^2, k = 0, 1, \dots\}.$$



Solutions for Even Order Nonlinear Boundary Value Problems with Constant Sign Green's Functions

In this chapter, as in the following ones, we will work with nonlinear problems. Such study could be (wrongly) considered as something completely independent from the study of the linear ones. However, as we will see in the remaining part of this Thesis, the properties satisfied by the Green's function are the basic tool to prove many existence results for nonlinear problems. This clearly justifies the necessity of doing a careful study of linear problems and, in particular, of the Green's functions.

In this chapter, we will show how to apply the results proved in previous chapters to ensure the existence of solution of some nonlinear problems. To do this, we will use the relations found in Corollaries 2.5.1 and 2.5.3 and apply the method of lower and upper solutions, following the line of [25].

The method of lower and upper solutions is a classical tool for proving the existence of solutions of nonlinear boundary value problems. Roughly speaking, this method works as follows: the existence of a lower solution, α , and an upper solution β , which are well-ordered, implies the existence of a solution lying between them. This way, we have information not only about the existence of a solution, but also about its location.

The usual tool to derive an existence result consists in the construction of a modified problem that satisfies the two following properties:

- (1) The nonlinear part of the modified equation is bounded.
- (2) The nonlinear part of the modified equation coincides with the nonlinear part when the solution lies between the lower and upper solution.

Unfortunately, there is not a direct way of constructing the aforementioned pair of lower and upper solutions. This is the reason why different generalizations on the definitions of lower and upper solutions have been considered in the literature, trying to weaken the regularity conditions that these functions must satisfy.

We include now some references that the reader can consult to find more information about this theory in a more general framework. First, we refer to the monograph [9], where the authors develop the classical theory of lower and upper solutions. Moreover, some recent results and open problems can be found in the works of Mawhin [109–112] and in the surveys of Cabada [17] and De Coster and Habets [44, 45], together with their monograph [46], and the references therein.

The novelty in our approach with respect to others presented in the literature is that we are able to ensure the existence of solution of a problem by means of lower and upper solutions of another problem with different boundary conditions. To the best of our knowledge, this approach is new in the literature.

All the results in this chapter are collected in [31].

In particular, we will consider nonlinear problems that fulfil the following scheme

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X, \quad (4.0.1)$$

being L the $2n$ -th order general linear operator defined in (2.1.1), namely

$$\begin{aligned} L u(t) \equiv & u^{(2n)}(t) + a_{2n-1}(t) u^{(2n-1)}(t) + \cdots + a_1(t) u'(t) \\ & + a_0(t) u(t), \quad t \in I, \end{aligned}$$

with $a_k : I \rightarrow \mathbb{R}$, $a_k \in L^\alpha(I)$, $\alpha \geq 1$, $k = 0, \dots, 2n - 1$.

We will assume that L is nonresonant on X , where, as in previous chapters, $X \subset W^{2n,1}(I)$ is a Banach space where the boundary conditions are included.

Finally, we shall mention that the constant sign of the Green's function will be a basic assumption to prove the existence of solutions.

This chapter is organized in the following way: First, Section 4.1 compiles some basic definitions and conditions that will be used throughout the remaining of the chapter. Section 4.2 includes the results proving the existence of solutions via lower and upper solutions method. Finally, Section 4.3 provides an example in which we prove the existence of solutions of the Dirichlet problem via lower and upper solutions of Neumann problem.

4.1. Preliminaries

It is clear that solutions of problem (4.0.1) correspond with the fixed points in X of the following integral operator

$$L^{-1} u(t) = \int_0^T G[T](t, s) f(s, u(s)) \, ds.$$

In particular, when the Banach space X is either $X_{N,T}$, $X_{D,T}$, $X_{M_1,T}$ or $X_{M_2,T}$ (defined as in Chapter 2), we obtain, respectively, the following nonlinear problems:

- Neumann problem:

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X_{N,T}, \quad (4.1.1)$$

- Dirichlet problem:

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X_{D,T}, \quad (4.1.2)$$

- Mixed problem 1:

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X_{M_1,T}, \quad (4.1.3)$$

- Mixed problem 2:

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X_{M_2,T}; \quad (4.1.4)$$

each of them with its corresponding equivalent integral operator:

$$T_N u(t) = \int_0^T G_N[T](t, s) f(s, u(s)) \, ds,$$

$$T_D u(t) = \int_0^T G_D[T](t, s) f(s, u(s)) \, ds,$$

$$T_{M_1} u(t) = \int_0^T G_{M_1}[T](t, s) f(s, u(s)) \, ds,$$

$$T_{M_2} u(t) = \int_0^T G_{M_2}[T](t, s) f(s, u(s)) \, ds.$$

Notation 4.1.1. Note that, as in Chapters 2 and 3, we will use the notation $G[T]$ to refer to the Green's function related to operator L .

Moreover, analogously to what we have done in Chapter 3, we will consider the parametrized operator defined from L , namely $L[\lambda] u(t) \equiv L u(t) + \lambda u(t)$. In this case, to stress also its dependence on λ , we will denote by $G[\lambda, T]$ the Green's function related to $L[\lambda]$.

For the purpose of finding fixed points of the previously defined integral operators, we shall use the following definitions.

Definition 4.1.2. We say that a function $\alpha \in X$ is a lower solution of problem (4.0.1) if

$$L\alpha(t) \geq f(t, \alpha(t)) \quad \text{for a. e. } t \in I.$$

Analogously, a function $\beta \in X$ is said to be an upper solution of problem (4.0.1) if

$$L\beta(t) \leq f(t, \beta(t)) \quad \text{for a. e. } t \in I.$$

Previous definitions are adapted to each of the considered problems by simply changing X by any of the suitable Banach spaces $X_{N,T}$, $X_{D,T}$, $X_{M_1,T}$ or $X_{M_2,T}$.

Before proving existence results for some of the problems, we will consider some conditions that will be used in the remainder of the chapter.

First, we will ask the nonlinearity f to satisfy the following property:

(L_0) The function $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ is a L^1 -Carathéodory function, that is,

- $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a. e. $t \in I$.
- For every $R > 0$ there exists $\varphi_R \in L^1(I)$ such that

$$|f(t, x)| \leq \varphi_R(t),$$

for all $x \in [-R, R]$ and a. e. $t \in I$.

Moreover, given two continuous functions α and β , we will state the following conditions:

(L_1) There exists some $\lambda \in \mathbb{R}$ such that for every $t \in I$ and $x \in [\alpha(t), \beta(t)]$, it holds that

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq f(t, x) + \lambda x \geq f(t, \beta(t)) + \lambda \beta(t),$$

and

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq 0 \geq f(t, \beta(t)) + \lambda \beta(t).$$

(L_2) There exists some $\lambda \in \mathbb{R}$ such that for every $t \in I$ and $x \in [\beta(t), \alpha(t)]$, it holds that

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq f(t, x) + \lambda x \geq f(t, \beta(t)) + \lambda \beta(t),$$

and

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq 0 \geq f(t, \beta(t)) + \lambda \beta(t).$$

4.2. Results of Existence of Solutions

In this section we will use the lower and upper solutions methods to prove the existence of solutions of the considered problems.

Theorem 4.2.1. *Assume that condition (L_0) holds and let α and β be lower and upper solutions of the Neumann problem (4.1.1), respectively, such that*

$$\alpha(t) \leq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_P[\lambda, 2T] \leq 0$ on $J \times J$, $G_D[\lambda, T] \leq 0$ on $I \times I$ and (L_1) holds. Then, there exists a solution u of the Dirichlet problem (4.1.2) such that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \text{for all } t \in I.$$

Proof. Let λ be such that $G_P[\lambda, 2T] \leq 0$ on $J \times J$, $G_D[\lambda, T] \leq 0$ on $I \times I$ and condition (L_1) holds. Consider the problem

$$L[\lambda] u(t) = f(t, u(t)) + \lambda u(t), \quad t \in I, \quad u \in X_{D,T}, \quad (4.2.1)$$

with $L[\lambda] u(t) \equiv L u(t) + \lambda u(t)$.

As a consequence, the solutions of problem (4.2.1) coincide with the solutions of (4.1.2). Also, these solutions correspond with fixed points of the following integral operator

$$T_D[\lambda] u(t) = \int_0^T G_D[\lambda, T](t, s) (f(s, u(s)) + \lambda u(s)) \, ds.$$

We will divide the proof into several steps. In particular, Steps 1 to 3 follow standard techniques but we include them for the sake of completeness.

Step 1: $T_D[\lambda]: \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is well-defined:

Let $u \in \mathcal{C}(I)$ and $(t_n)_{n \in \mathbb{N}} \subset I$ such that $\lim_{n \rightarrow \infty} t_n = t_0 \in I$. On the one hand, from property $(G2)$ in the definition of Green's function (Definition 1.1.1), $G_D[\lambda, T](\cdot, s)$ is uniformly continuous on I . Thus,

$$\lim_{n \rightarrow \infty} G_D[\lambda, T](t_n, s) (f(s, u(s)) + \lambda u(s)) = G_D[\lambda, T](t_0, s) (f(s, u(s)) + \lambda u(s)).$$

On the other hand it holds that

$$|G_D[\lambda, T](t, s) (f(s, u(s)) + \lambda u(s))| \leq |G_D[\lambda, T](t, s)| (\varphi_{\|u\|}(s) + \lambda \|u\|),$$

where $\|u\|$ denotes the usual supremum norm. Moreover, from (G2) in Definition 1.1.1, $G_D[\lambda, T]$ is continuous on $I \times I$ and so it is bounded on $I \times I$ by some constant M . Therefore

$$|G_D[\lambda, T](t, s) (f(s, u(s)) + \lambda u(s))| \leq M (\varphi_{\|u\|}(s) + \lambda \|u\|), \text{ a. e. } s \in I$$

and, since the right hand side of previous inequality is in $L^1(I)$ by Lebesgue's Dominated Convergence Theorem we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} T_D[\lambda] u(t_n) &= \lim_{n \rightarrow \infty} \int_0^T G_D[\lambda, T](t_n, s) (f(s, u(s)) + \lambda u(s)) \, ds \\ &= \int_0^T \lim_{n \rightarrow \infty} G_D[\lambda, T](t_n, s) (f(s, u(s)) + \lambda u(s)) \, ds \\ &= \int_0^T G_D[\lambda, T](t_0, s) (f(s, u(s)) + \lambda u(s)) \, ds = T_D[\lambda] u(t_0). \end{aligned}$$

Thus, $T_D[\lambda] u \in \mathcal{C}(I)$.

Step 2: $T_D[\lambda]$ is continuous:

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence which converges to u in $\mathcal{C}(I)$. Then, there exists some $R \in \mathbb{R}^+$ such that $\|u_n\| \leq R$ for all $n \in \mathbb{N}$.

Now, on the one hand, from (L_0) , we deduce that

$$\lim_{n \rightarrow \infty} f(s, u_n(s)) + \lambda u_n(s) = f(s, u(s)) + \lambda u(s), \text{ for a. e. } s \in I.$$

On the other hand,

$$|G_D[\lambda, T](t, s) (f(s, u_n(s)) + \lambda u_n(s))| \leq M (\varphi_R(s) + \lambda R), \text{ a. e. } s \in I$$

and, since the right hand side of previous inequality is in $L^1(I)$ by Lebesgue's Dominated Convergence Theorem we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} T_D[\lambda] u_n(t) &= \lim_{n \rightarrow \infty} \int_0^T G_D[\lambda, T](t, s) (f(s, u_n(s)) + \lambda u_n(s)) \, ds \\ &= \int_0^T \lim_{n \rightarrow \infty} G_D[\lambda, T](t, s) (f(s, u_n(s)) + \lambda u_n(s)) \, ds \\ &= \int_0^T G_D[\lambda, T](t, s) (f(s, u(s)) + \lambda u(s)) \, ds = T_D[\lambda] u(t). \end{aligned}$$

Thus we can conclude that operator $T_D[\lambda]$ is continuous.

Step 3: $T_D[\lambda]$ is a compact operator:

Take

$$B = \{u \in \mathcal{C}(I); \|u\| < r\}.$$

First, we will prove that $T_D[\lambda](B)$ is uniformly bounded:

$$\begin{aligned} \|T_D[\lambda] u\| &= \sup_{t \in I} \left| \int_0^T G_D[\lambda, T](t, s) (f(s, u(s)) + \lambda u(s)) \, ds \right| \\ &\leq \int_0^T M (\varphi_r(s) + \lambda r) \, ds = M \left(\int_0^T \varphi_r(s) \, ds + \lambda r T \right) \end{aligned}$$

and, since $\varphi_r \in L^1(I)$, it is clear that $T_D[\lambda](B)$ is uniformly bounded.

Now, we will prove that $T_D[\lambda](B)$ is equicontinuous. We have that

$$\begin{aligned} &|(T_D[\lambda] u)(t_1) - (T_D[\lambda] u)(t_2)| \\ &\leq \int_0^T |G_D[\lambda, T](t_1, s) - G_D[\lambda, T](t_2, s)| (|f(s, u(s))| + \lambda |u(s)|) \, ds \\ &\leq \int_0^T |G_D[\lambda, T](t_1, s) - G_D[\lambda, T](t_2, s)| (\varphi_r(s) + \lambda r) \, ds \end{aligned}$$

and, since $G_D[\lambda, T]$ is uniformly continuous on $I \times I$, it occurs that for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|t_1 - t_2| < \delta$,

$$|(T_D[\lambda] u)(t_1) - (T_D[\lambda] u)(t_2)| \leq \varepsilon \int_0^T (\varphi_r(s) + \lambda r) \, ds.$$

Thus, the fact that $\varphi_r \in L^1(I)$ lets us conclude that $T_D[\lambda](B)$ is equicontinuous.

As a consequence, by Ascoli-Arzelà's Theorem (Theorem 1.2.2), we deduce that $T_D[\lambda](B)$ is relatively compact in $\mathcal{C}(I)$ and thus $T_D[\lambda]$ is a compact operator.

Step 4: $\alpha \leq T_D[\lambda] \alpha$ and $\beta \geq T_D[\lambda] \beta$ on I .

From Corollary 2.5.1, we know that $G_P[\lambda, 2T] \leq 0$ on $J \times J$ implies that

$$G_N[\lambda, T](t, s) \leq -|G_D[\lambda, T](t, s)|, \quad \text{for all } t, s \in I. \quad (4.2.2)$$

On the other hand, the fact that $\alpha \in X_{N,T} \subset W^{2n,1}(I)$ and $L\alpha(t) \geq f(t, \alpha(t))$ for a. e. $t \in I$ means that there exists a nonnegative function $g \in L^1(I)$, such that

$$L\alpha(t) + \lambda \alpha(t) = f(t, \alpha(t)) + \lambda \alpha(t) + g(t), \quad \text{for a. e. } t \in I.$$

Therefore, since $\alpha \in X_{N,T}$, it holds that

$$\alpha(t) = \int_0^T G_N[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds + \int_0^T G_N[\lambda, T](t, s) g(s) \, ds.$$

From (4.2.2), it is deduced that $G_N[\lambda, T]$ is nonpositive. Thus, $G_N[\lambda, T](t, s)g(s)$ is nonpositive for a.e. $t, s \in I$ and so, the second integral in previous expression is less or equal than zero.

Moreover, we also deduce from (4.2.2) that

$$G_N[\lambda, T] \leq G_D[\lambda, T] \quad \text{on } I \times I.$$

Therefore, taking into account the fact that (from (L_1)) $f(s, \alpha(s)) + \lambda \alpha(s) \geq 0$ for a.e. $s \in I$, we obtain the following inequalities for all $t \in I$:

$$\begin{aligned} \alpha(t) &\leq \int_0^T G_N[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds \\ &\leq \int_0^T G_D[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds = T_D[\lambda] \alpha(t). \end{aligned}$$

Analogously, from $\beta \in W^{2n,1}(I)$ and $L\beta(t) \leq f(t, \beta(t))$, we deduce that there exists a nonpositive function $h \in L^1(I)$, such that

$$L\beta(t) + \lambda \beta(t) = f(t, \beta(t)) + \lambda \beta(t) + h(t), \quad \text{for a.e. } t \in I.$$

As a consequence, reasoning analogously to the previous case and taking into account that $f(s, \beta(s)) + \lambda \beta(s) \leq 0$ for a.e. $s \in I$, we obtain the following inequalities

$$\begin{aligned} \beta(t) &= \int_0^T G_N[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds + \int_0^T G_N[\lambda, T](t, s) h(s) \, ds \\ &\geq \int_0^T G_N[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds \\ &\geq \int_0^T G_D[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds = T_D[\lambda] \beta(t). \end{aligned}$$

Step 5: $T_D[\lambda](\alpha, \beta) \subset [\alpha, \beta]$, where

$$[\alpha, \beta] \equiv \{u \in \mathcal{C}(I) : \alpha(t) \leq u(t) \leq \beta(t), \text{ for all } t \in I\}.$$

We will decompose operator $T_D[\lambda]$ as a composition of two operators. First, consider the Nemytskii operator $N[\lambda]: \mathcal{C}(I) \rightarrow L^1(I)$ defined in the following way

$$N[\lambda] u(t) = f(t, u(t)) + \lambda u(t), \quad \text{for a.e. } t \in I.$$

On the other hand, consider operator $K[\lambda]: L^1(I) \rightarrow \mathcal{C}(I)$ defined as

$$K[\lambda] \sigma(t) = \int_0^T G_D[\lambda, T](t, s) \sigma(s) \, ds, \quad \text{for all } t \in I.$$

It is clear that

$$T_D[\lambda] = K[\lambda] \circ N[\lambda].$$

Moreover, let's see that operator $K[\lambda]$ is nonincreasing in $[\alpha, \beta]$. Indeed, take $\sigma_1, \sigma_2 \in L^1(I)$ such that $\sigma_1(t) \leq \sigma_2(t)$ for a.e. $t \in I$. Then, since $G_D[\lambda, T]$ is nonpositive, it holds that

$$G_D[\lambda, T](t, s) \sigma_1(s) \geq G_D[\lambda, T](t, s) \sigma_2(s), \quad \text{for a.e. } t, s \in I$$

and, therefore,

$$\begin{aligned} K[\lambda] \sigma_1(t) &= \int_0^T G_D[\lambda, T](t, s) \sigma_1(s) \, ds \geq \int_0^T G_D[\lambda, T](t, s) \sigma_2(s) \, ds \\ &= K[\lambda] \sigma_2(t), \quad \text{for all } t \in I. \end{aligned}$$

Now, let $u \in [\alpha, \beta]$. From (L_1) we have that

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq f(t, u(t)) + \lambda u(t) \geq f(t, \beta(t)) + \lambda \beta(t), \quad \text{a.e. } t \in I$$

and so

$$\alpha(t) \leq T_D[\lambda] \alpha(t) \leq T_D[\lambda] u(t) \leq T_D[\lambda] \beta(t) \leq \beta(t), \quad \forall t \in I.$$

We conclude that $T_D[\lambda] u \in [\alpha, \beta]$ for all $u \in [\alpha, \beta]$.

Step 6: Operator $T_D[\lambda]$ has a fixed point in $X_D \cap [\alpha, \beta]$.

Since the interval $[\alpha, \beta]$ is a closed, convex, bounded and nonempty subset of the Banach space X , $T_D[\lambda]$ is a compact operator and $T_D[\lambda](\alpha, \beta) \subset [\alpha, \beta]$, then we are in the suitable conditions to apply Schauder's fixed point Theorem (Theorem 1.2.3) which ensures us the existence of a fixed point of $T_D[\lambda]$ on $[\alpha, \beta]$. Obviously, this fixed point satisfies Dirichlet boundary conditions and therefore it is a solution of problem (4.1.2). \square

Remark 4.2.2. Note that the functions α and β considered in previous theorem are not required to belong to $X_{D,T}$, that is, they may not be lower and upper solutions of Dirichlet problem, that is, the equalities $u^{(2k)}(0) = 0$ and $u^{(2k)}(T) = 0$ may fail for some values of k .

In an analogous way, we can prove the following result when $G_P[\lambda, 2T]$ is non-negative and hypothesis (L_2) holds.

Theorem 4.2.3. Assume that condition (L_0) holds and let α and β be lower and upper solutions of Neumann problem (4.1.1), respectively, such that

$$\alpha(t) \geq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_P[\lambda, 2T] \geq 0$ on $J \times J$, $G_D[\lambda, T] \geq 0$ on $I \times I$ and (L_2) is satisfied. Then, there exists a solution u of the Dirichlet problem (4.1.2) such that

$$\beta(t) \leq u(t) \leq \alpha(t), \quad \text{for all } t \in I.$$

Proof. The proof is analogous to the one of Theorem 4.2.1, so we will only detail the parts of it which present some differences.

Let λ be such that $G_P[\lambda, 2T] \geq 0$ on $J \times J$, $G_D[\lambda, T] \geq 0$ on $I \times I$ and condition (L_2) holds. Consider operator $T_D[\lambda]$ as defined in the proof of Theorem 4.2.1.

Step 1: $T_D[\lambda]: \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is well-defined, continuous and compact:

The proof is exactly the same than in Theorem 4.2.1.

Step 2: $\alpha \geq T_D[\lambda] \alpha$ and $\beta \leq T_D[\lambda] \beta$.

From Corollary 2.5.1, we know that $G_P[\lambda, 2T] \leq 0$ on $J \times J$ implies that

$$G_N[\lambda, T](t, s) \geq |G_D[\lambda, T](t, s)| = G_D[\lambda, T](t, s), \quad \text{for all } t, s \in I. \quad (4.2.3)$$

In this case, there exists a nonnegative function $g \in L^1(I)$ such that

$$L \alpha(t) + \lambda \alpha(t) = f(t, \alpha(t)) + \lambda \alpha(t) + g(t), \quad \text{for a.e. } t \in I$$

and so the following inequalities hold for all $t \in I$:

$$\begin{aligned} \alpha(t) &= \int_0^T G_N[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds + \int_0^T G_N[\lambda, T](t, s) g(s) \, ds \\ &\geq \int_0^T G_N[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds \\ &\geq \int_0^T G_D[\lambda, T](t, s) (f(s, \alpha(s)) + \lambda \alpha(s)) \, ds = T_D[\lambda] \alpha(t). \end{aligned}$$

Analogously, there exists a nonpositive function $h \in L^1(I)$ such that

$$L \beta(t) + \lambda \beta(t) = f(t, \beta(t)) + \lambda \beta(t) + h(t), \quad \text{for a.e. } t \in I$$

and

$$\begin{aligned} \beta(t) &= \int_0^T G_N[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds + \int_0^T G_N[\lambda, T](t, s) h(s) \, ds \\ &\leq \int_0^T G_N[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds \\ &\leq \int_0^T G_D[\lambda, T](t, s) (f(s, \beta(s)) + \lambda \beta(s)) \, ds = T_D[\lambda] \beta(t). \end{aligned}$$

Step 3: $T_D[\lambda]([\beta, \alpha]) \subset [\beta, \alpha]$.

In this case, operator $K[\lambda]$ is nondecreasing.

Let $u \in [\beta, \alpha]$. From (L_2) we have that

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq f(t, u(t)) + \lambda u(t) \geq f(t, \beta(t)) + \lambda \beta(t), \quad \text{a. e. } t \in I$$

and so

$$\alpha(t) \geq T_D[\lambda] \alpha(t) \geq T_D[\lambda] u(t) \geq T_D[\lambda] \beta(t) \geq \beta(t), \quad \forall t \in I$$

and we conclude that $T_D[\lambda] u \in [\beta, \alpha]$ for all $u \in [\beta, \alpha]$.

Step 4: Operator $T_D[\lambda]$ has a fixed point in $X_D \cap [\beta, \alpha]$.

Analogously to the proof of Theorem 4.2.1, this fact is deduced from Schauder's fixed point Theorem. \square

Remark 4.2.4. We must note that when $G_D[\lambda, T]$ has constant sign, there exist α and β , lower and upper solutions of Dirichlet problem, respectively, and it is satisfied that

$$f(t, \alpha(t)) + \lambda \alpha(t) \geq f(t, x) + \lambda x \geq f(t, \beta(t)) + \lambda \beta(t)$$

for every $t \in I$ and $x \in [\alpha(t), \beta(t)]$, then there exists a solution of the Dirichlet problem (4.1.2) (see, for instance, [25] for the case $2n = 4$). In this case, by adding the hypotheses on the sign of $f(t, \alpha(t)) + \lambda \alpha(t)$ and $f(t, \beta(t)) + \lambda \beta(t)$, we can ensure the existence of a solution for problem (4.1.2) when we have lower and upper solutions of Neumann problem (4.1.1).

Now, using the inequalities in Corollary 2.5.3, we can obtain similar results to prove the existence of solutions of Mixed 1 and Dirichlet problems.

Theorem 4.2.5. Assume that condition (L_0) holds and let α and β be lower and upper solutions of the Neumann problem (4.1.1), respectively, such that

$$\alpha(t) \leq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_N[\lambda, 2T] \leq 0$ on $J \times J$, $G_{M_1}[\lambda, T] \leq 0$ on $I \times I$ and (L_1) holds. Then, there exists a solution u of the Mixed problem 1 (4.1.3) such that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \text{for all } t \in I.$$

Theorem 4.2.6. *Assume that condition (L_0) holds and let α and β be lower and upper solutions of the Neumann problem (4.1.1), respectively, such that*

$$\alpha(t) \geq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_N[\lambda, 2T] \geq 0$ on $J \times J$, $G_{M_1}[\lambda, T] \geq 0$ on $I \times I$ and (L_2) holds. Then, there exists a solution u of the Mixed problem 1 (4.1.3) such that

$$\beta(t) \leq u(t) \leq \alpha(t), \quad \text{for all } t \in I.$$

Theorem 4.2.7. *Assume that condition (L_0) holds and let α and β be lower and upper solutions of Mixed problem 2 (4.1.4), respectively, such that*

$$\alpha(t) \leq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_D[\lambda, 2T] \leq 0$ on $J \times J$, $G_D[\lambda, T] \leq 0$ on $I \times I$ and (L_1) holds. Then, there exists a solution u of the Dirichlet problem (4.1.2) such that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \text{for all } t \in I.$$

Theorem 4.2.8. *Assume that condition (L_0) holds and let α and β be lower and upper solutions of Mixed problem 2 (4.1.4), respectively, such that*

$$\alpha(t) \geq \beta(t) \quad \text{for all } t \in I.$$

Moreover, assume that there exists some λ for which $G_D[\lambda, 2T] \geq 0$ on $J \times J$, $G_D[\lambda, T] \geq 0$ on $I \times I$ and (L_2) holds. Then, there exists a solution u of the Dirichlet problem (4.1.2) such that

$$\beta(t) \leq u(t) \leq \alpha(t), \quad \text{for all } t \in I.$$

4.2.1. Particular Case: Second Order Problem

We shall briefly comment in this subsection some particularities which may occur when dealing with the case $n = 1$ but are not true, in general, for any arbitrary n .

In particular, when considering Definition 4.1.2, for the general case it is required that both the lower and the upper solution belong to the Banach spaces $X_{N,T}$, $X_{D,T}$, $X_{M_1,T}$ or $X_{M_2,T}$. For the case $n = 1$, it is possible to weaken these definitions in the way that it is shown below.

Definition 4.2.9. Assume that $n = 1$. Then, a function $\alpha \in W^{2,1}(I)$ is said to be a lower solution of the Neumann problem (4.1.1) if

$$\begin{cases} L\alpha(t) \geq f(t, \alpha(t)), & t \in I, \\ \alpha'(0) \geq 0, & \alpha'(T) \leq 0. \end{cases}$$

Analogously, a function $\beta \in W^{2,1}(I)$ is called an upper solution of the Neumann problem (4.1.1) if

$$\begin{cases} L\beta(t) \leq f(t, \beta(t)), & t \in I, \\ \beta'(0) \leq 0, & \beta'(T) \geq 0. \end{cases}$$

Definition 4.2.10. Assume that $n = 1$. Then, a function $\alpha \in W^{2,1}(I)$ is said to be a lower solution of the Dirichlet problem (4.1.2) if

$$\begin{cases} L\alpha(t) \geq f(t, \alpha(t)), & t \in I, \\ \alpha(0) \leq 0, & \alpha(T) \leq 0. \end{cases}$$

Analogously, a function $\beta \in W^{2,1}(I)$ is called an upper solution of the Dirichlet problem (4.1.2) if

$$\begin{cases} L\beta(t) \leq f(t, \beta(t)), & t \in I, \\ \beta(0) \geq 0, & \beta(T) \geq 0. \end{cases}$$

Definition 4.2.11. Assume that $n = 1$. Then, a function $\alpha \in W^{2,1}(I)$ is said to be a lower solution of the Mixed problem 1 (4.1.3) if

$$\begin{cases} L\alpha(t) \geq f(t, \alpha(t)), & t \in I, \\ \alpha'(0) \geq 0, & \alpha(T) \leq 0. \end{cases}$$

Analogously, a function $\beta \in W^{2,1}(I)$ is called an upper solution of the Mixed 1 problem (4.1.3) if

$$\begin{cases} L\beta(t) \leq f(t, \beta(t)), & t \in I, \\ \beta'(0) \leq 0, & \beta(T) \geq 0. \end{cases}$$

Definition 4.2.12. Assume that $n = 1$. Then, a function $\alpha \in W^{2,1}(I)$ is said to be a lower solution of the Mixed problem 2 (4.1.4) if

$$\begin{cases} L\alpha(t) \geq f(t, \alpha(t)), & t \in I, \\ \alpha(0) \leq 0, & \alpha'(T) \leq 0. \end{cases}$$

Analogously, a function $\beta \in W^{2,1}(I)$ is called an upper solution of the Mixed 2 problem (4.1.4) if

$$\begin{cases} L\beta(t) \leq f(t, \beta(t)), & t \in I, \\ \beta(0) \geq 0, & \beta'(T) \geq 0. \end{cases}$$

Apart from these generalizations with respect to the definitions of lower and upper solutions, some details need also be changed in the theorems of existence of solution when $n = 1$ and $a_1 \equiv 0$. In particular, as a consequence of Theorem 3.2.22, it holds that the constant sign of $G_P[\lambda, 2T]$ implies that $G_D[\lambda, T]$ is nonpositive, so this hypothesis can be eliminated from Theorem 4.2.1. The same way, the constant sign of $G_N[\lambda, 2T]$ implies that $G_{M_1}[\lambda, T]$ is nonpositive and we can remove this hypothesis from Theorem 4.2.5. Finally, the hypothesis that $G_D[\lambda, T]$ is nonpositive can also be eliminated from Theorem 4.2.7 as it can be deduced from the constant sign of $G_D[\lambda, 2T]$.

Furthermore, due to these relations between the constant sign of different Green's functions, Theorems 4.2.3, 4.2.6 and 4.2.8 do not make sense for the case $n = 1$ as their hypotheses are never fulfilled in such case.

4.3. An Example

We will see in this section an example in which Theorem 4.2.1 can be applied.

Example 4.3.1. Consider the following nonlinear Dirichlet problem on $[0, 1]$

$$\begin{cases} u^{(4)}(t) + u(t) = t^2 \left(\frac{1}{10} + \arctan(u(t)) \right), & t \in [0, 1], \\ u(0) = u(1) = 0, & u''(0) = u''(1) = 0. \end{cases} \quad (4.3.1)$$

Using [24], we can calculate the Green's function related to the periodic problem on $[0, 2]$

$$\begin{cases} u^{(4)}(t) + u(t) = 0, & t \in [0, 2], \\ u(0) = u(1), & u'(0) = u'(1), \quad u''(0) = u''(1), \quad u'''(0) = u'''(1), \end{cases} \quad (4.3.2)$$

which is nonnegative on $[0, 2] \times [0, 2]$. This function is represented in Figure 4.3.1.

The same way, using [24], we can calculate the Green's function related to the homogeneous Dirichlet problem

$$\begin{cases} u^{(4)}(t) + u(t) = 0, & t \in [0, 1], \\ u(0) = u(1) = 0, & u''(0) = u''(1) = 0, \end{cases} \quad (4.3.3)$$

which is also nonnegative on $[0, 1] \times [0, 1]$. This function is represented in Figure 4.3.2.

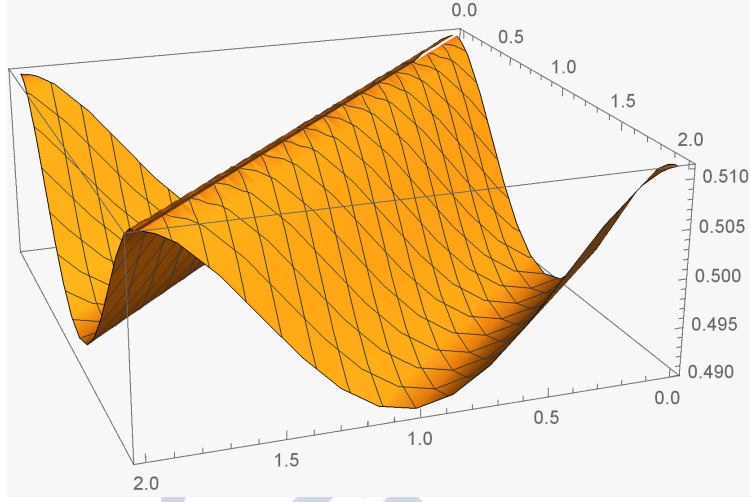


Figure 4.3.1: Green's function related to the periodic problem (4.3.2).

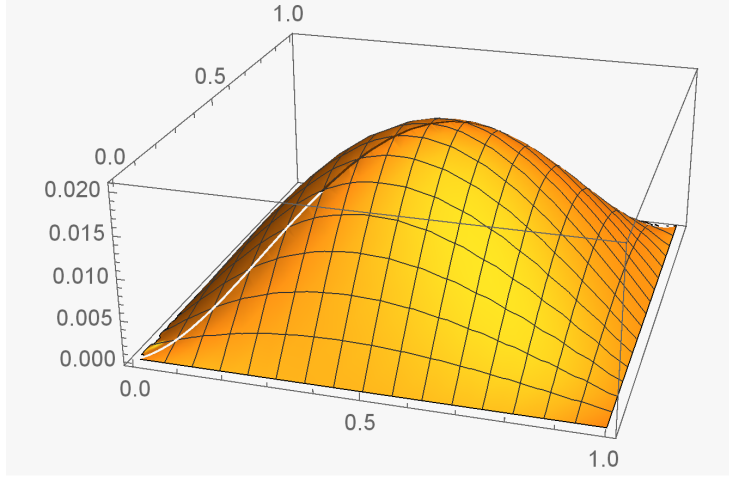


Figure 4.3.2: Green's function related to the Dirichlet problem (4.3.3).

Observe that, with the notation given in Theorem 4.2.3, we are choosing $\lambda = 0$. Now, we will see that $\alpha(t) = 1$ and $\beta(t) = -1$ are lower and upper solutions, respectively, of the Neumann problem

$$\begin{cases} u^{(4)}(t) + u(t) = t^2 \left(\frac{1}{10} + \arctan(u(t)) \right), & t \in [0, 1], \\ u'(0) = u'(1) = 0, & u'''(0) = u'''(1) = 0. \end{cases}$$

Indeed, for all $t \in [0, 1]$,

$$\begin{cases} \alpha^{(4)}(t) + \alpha(t) = 1 \geq t^2 \left(\frac{1}{10} + \arctan(\alpha(t)) \right) = \left(\frac{1}{10} + \frac{\pi}{4} \right) t^2, \\ \alpha'(0) = \alpha'(1) = 0, \quad \alpha'''(0) = \alpha'''(1) = 0. \end{cases}$$

and

$$\begin{cases} \beta^{(4)}(t) + \beta(t) = -1 \leq t^2 \left(\frac{1}{10} + \arctan(\beta(t)) \right) = - \left(-\frac{1}{10} + \frac{\pi}{4} \right) t^2, \\ \beta'(0) = \beta'(1) = 0, \quad \beta'''(0) = \beta'''(1) = 0. \end{cases}$$

Moreover, it holds that for $x \in [-1, 1]$, $\arctan(x) \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$ and so

$$\left(\frac{1}{10} + \frac{\pi}{4} \right) t^2 \geq \left(\frac{1}{10} + \arctan(x) \right) t^2 \geq \left(\frac{1}{10} - \frac{\pi}{4} \right) t^2.$$

Finally, we have that, for all $t \in [0, 1]$,

$$f(t, \alpha(t)) = \left(\frac{1}{10} + \frac{\pi}{4} \right) t^2 \geq 0$$

and

$$f(t, \beta(t)) = \left(\frac{1}{10} - \frac{\pi}{4} \right) t^2 \leq 0.$$

Thus, hypothesis (L_2) is satisfied.

Therefore, we are in conditions to apply Theorem 4.2.3 which guarantees the existence of a solution u of the Dirichlet problem (4.3.1) such that

$$-1 \leq u(t) \leq 1, \quad \text{for all } t \in [0, 1].$$

Note that α and β are not lower and upper solutions of the Dirichlet problem as they are defined in Definition 4.1.2.

Positive Solutions for Nonlinear Second Order Boundary Value Problems with Sign-Changing Green's Functions

5.1. Introduction

In Chapter 4 we have proved the existence of solution of nonlinear boundary value problems in the case where the Green's functions have constant sign. To do this, we have used the method of lower and upper solutions.

Another standard technique when dealing with nonlinear boundary value problems consists in obtaining the existence of positive solutions through Krasnosel'skii's fixed point Theorem on cones, or fixed point index theory. In these cases, the positivity of the associated Green's functions is usually a fundamental tool to prove such results. However, in this chapter, using the aforementioned technique, we will be able to guarantee the existence of positive solutions for several problems in which the Green's function changes its sign on the square of definition.

In particular, we will prove such results for boundary value problems related to the Hill's operator (which has already been considered in Chapter 3). To do this, some of the properties satisfied by the Green's function which were proved in Chapter 3 will be the key points to prove some of the main results. This shows, once more, the importance of studying the properties of linear problems (and, specially, Green's functions) before dealing with nonlinear ones.

As we will see, the hypotheses that the nonlinear part f must satisfy in this chapter will be stronger than in Chapter 4. Nevertheless, the results obtained will also be more powerful as, contrary to Chapter 4, we will allow the Green's function to change its sign and we will be able to ensure now the positivity of the solutions.

Positivity results for boundary value problems where the Green's function can vanish are treated for example in [63, 147]. In [63], Graef, Kong and Wang studied

the periodic boundary value problem (with $T = 1$ in the paper)

$$\begin{cases} u''(t) + a(t)u(t) = g(t)f(u(t)), & t \in (0, T), \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

with f and g nonnegative continuous functions and g satisfying the condition

$$\min_{t \in [0, T]} g(t) > 0.$$

Moreover, they assumed the Green's function to be nonnegative and to satisfy the following condition:

$$\min_{0 \leq s \leq T} \int_0^T G(t, s) dt > 0. \quad (5.1.1)$$

We note that the method in the aforementioned reference can not be used for Dirichlet and Mixed problems, as their related Green's functions do not satisfy condition (5.1.1).

In [147], Webb considered weaker assumptions to prove the existence of positive solutions of the previous problem, but he still assumed the Green's function to be nonnegative. Despite our results do not require the Green's function to be nonnegative, as we will see, they could be applied to this particular case, obtaining positive solutions assuming an integral condition weaker than (5.1.1) (see Remarks 5.3.6 and 5.3.11 in Section 5.3).

On the other hand, some existence results for boundary value problems with sign-changing Green's function were considered in [28, 79], where the authors asked for the existence of a subinterval $[c, d] \subset [0, T]$, a function $\phi \in L^1([0, T])$ and a constant $c \in (0, 1]$ such that the Green's function G satisfies the following condition:

$$\begin{aligned} |G(t, s)| &\leq \phi(s) \text{ for all } t \in [0, T] \text{ and almost every } s \in [0, T], \\ G(t, s) &\geq c\phi(s) \text{ for all } t \in [c, d] \text{ and almost every } s \in [0, T]. \end{aligned} \quad (5.1.2)$$

It must be pointed out that, if we consider a periodic problem with constant potential $a(t) = \rho^2$ for which the related Green's function changes its sign (i.e. $\rho > \pi/T$, $\rho \neq 2k\pi/T$, $k = 1, 2, \dots$), condition (5.1.2) is never fulfilled for any strictly positive function ϕ . This is due to the fact that in such situation the Green's function is constant along the straight lines of slope equals to one (as we have seen in Lemma 2.4.3). On the other hand, as we will prove in Section 5.4, our results can be applied without further complications to this case.

Moreover, for Dirichlet boundary value problem with constant potential $a(t) = \rho^2$ with sign-changing Green's function (i.e. $\rho > \pi/T$, $\rho \neq k\pi/T$, $k = 1, 2, \dots$), as a direct consequence of expression (5.5.1) below, it is immediate to verify that condition (5.1.2) holds if and only if ρ^2 lies between the first and the second eigenvalues

of the problem ($\frac{\pi}{T} < \rho < \frac{2\pi}{T}$) but it is never satisfied for $\rho > \frac{2\pi}{T}$. However, as we will point out in Section 5.5, our results can be applied for any nonresonant value of $\rho > \pi/T$. Despite of this, we must note that the conditions are more restrictive when ρ increases.

Furthermore, in [28, 79] the authors proved the existence of solutions in the cone

$$K_0 = \left\{ u \in \mathcal{C}[0, T], \min_{t \in [c, d]} u(t) \geq c \|u\| \right\},$$

that is, they ensured the positivity of the solutions on the subinterval $[c, d]$ but such solutions were allowed to change sign when considering the whole interval $[0, T]$.

As far as we know, positive solutions for boundary value problems with sign-changing Green's function can be tracked only as back as 2011 in the papers [104, 163]. In the first of these papers, R. Ma considers the following one parameter family of problems:

$$\begin{cases} u''(t) + a(t)u(t) = \lambda g(t)f(u(t)), & t \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \quad (5.1.3)$$

By using the Schauder's fixed point Theorem, the author obtains the existence of a positive solution for sufficiently small values of λ . These existence results are not comparable with the ones we will obtain in this chapter.

In the second paper [163], S. Zhong and Y. An study the following autonomous periodic boundary value problem, with constant potential $\rho \in (0, \frac{3\pi}{2T}]$:

$$\begin{cases} u''(t) + \rho^2 u(t) = f(u(t)), & t \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \quad (5.1.4)$$

In this case, it is very well-known that the related Green's function $G_P(t, s) \geq 0$ for all $\rho \in (0, \frac{\pi}{T}]$ and it changes sign for $\rho \in (\frac{\pi}{T}, \frac{3\pi}{2T}]$ (see [16, 18]). With this, it can be defined the constant

$$\delta = \begin{cases} \infty, & \rho \in (0, \frac{\pi}{T}], \\ \inf_{t \in I} \frac{\int_0^T G_P^+(t, s) \, ds}{\int_0^T G_P^-(t, s) \, ds}, & \rho \in (\frac{\pi}{T}, \frac{3\pi}{2T}], \end{cases}$$

and using the Krasnoselskii's fixed point Theorem, the authors prove the following existence result.

Theorem 5.1.1 ([163, Theorem 3]). *Suppose that the following assumptions are fulfilled:*

(J1) $f : [0, \infty) \rightarrow [0, \infty)$ is continuous.

(J2) $0 \leq m = \inf_{u \geq 0} \{f(u)\}$ and $M = \sup_{u \geq 0} \{f(u)\} \leq M \leq \infty$.

(J3) $M/m \leq \delta$, with $M/m = \infty$ when $m = 0$.

Moreover, if $\delta = \infty$ assume that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} < \rho^2 < \lim_{x \rightarrow 0^+} \frac{f(x)}{x}.$$

Then problem (5.1.4) has a positive solution on $[0, T]$.

Concerning this specific case, we improve the range of the values ρ for which the result is still valid. Furthermore, we apply our study to nonconstant potentials and nonautonomous nonlinear parts.

As we will see, some of the positivity conditions imposed for the periodic boundary value problem cannot be adapted for Dirichlet problem, so the approach that must be used needs to be considerably modified, by using, in this case, a different type of cones.

This chapter is divided in the following way: in Section 5.2 we state some preliminary results considering the Hill's operator, in Section 5.3 some new results concerning the existence of a positive solution for the Hill's periodic problem in the case that the Green's function may change sign are proved. Moreover, in this section, such existence results are generalized to other boundary conditions. In Section 5.4 we improve Theorem 5.1.1 for the periodic problem with a constant potential and in Section 5.5 we approach the Dirichlet problem, also in the case of a constant potential, where as far as we know, no results for sign-changing Green's functions were proved before.

All the results in this chapter are compiled in [27].

5.2. Preliminaries

Consider the particular case of operator L defined in (2.1.1) for $n = 1$ and $a_1 \equiv 0$, that is, the Hill's operator related to the potential a

$$L u(t) \equiv u''(t) + a(t) u(t), \quad t \in I,$$

where $a : I \rightarrow \mathbb{R}$, $a \in L^\alpha(I)$, $\alpha \geq 1$.

As in previous chapters, we denote $x \succ 0$ on I if and only if $x \geq 0$ on I and $\int_0^T x(s) \, ds > 0$.

Since throughout this chapter we will always work on the same interval $[0, T]$, it is not necessary to stress the dependence of the problem on the parameter T . Therefore, when working both with Green's functions and eigenvalues, and contrary to what we did in the previous chapter, we will skip the indication about the parameter T . This way, we will denote by G_P , G_N , G_D , G_{M_1} and G_{M_2} the related Green's functions and λ_0^P , λ_0^N , λ_0^D , $\lambda_0^{M_1}$ and $\lambda_0^{M_2}$ the corresponding smallest eigenvalue of each of the problems (periodic, Neumann, Dirichlet, Mixed 1 and Mixed 2), all of them considered on the interval I . Analogously, λ_0^A will be the smallest eigenvalue of the anti-periodic problem.

For the reader's convenience, we rewrite now the following relations which have been proved in previous chapters and will be the key points to show some of the following results.

- Lemma 5.2.1.** 1. $G_N(t, s) < 0$ on $I \times I$ if and only if $\lambda_0^N > 0$.
2. $G_N(t, s) \geq 0$ on $I \times I$ if and only if $\lambda_0^N < 0$, $\lambda_0^{M_1} \geq 0$ and $\lambda_0^{M_2} \geq 0$.
3. G_N changes sign if and only if $\min\{\lambda_0^{M_1}, \lambda_0^{M_2}\} < 0$.
4. $G_D(t, s) < 0$ on $(0, T) \times (0, T)$ if and only if $\lambda_0^D > 0$.
5. G_D changes sign if and only if $\lambda_0^D < 0$.
6. $G_{M_1}(t, s) < 0$ on $[0, T] \times [0, T]$ if and only if $\lambda_0^{M_1} > 0$.
7. G_{M_1} changes sign if and only if $\lambda_0^{M_1} < 0$.
8. $G_{M_2}(t, s) < 0$ on $(0, T] \times (0, T]$ if and only if $\lambda_0^{M_2} > 0$.
9. G_{M_2} changes sign if and only if $\lambda_0^{M_2} < 0$.
10. $G_P(t, s) < 0$ on $I \times I$ if and only if $\lambda_0^P > 0$.
11. $G_P(t, s) \geq 0$ on $I \times I$ if and only if $\lambda_0^P < 0$, $\lambda_0^A \geq 0$.
12. G_P changes sign if and only if $\lambda_0^A < 0$.

5.3. Periodic Boundary Value Problems

Consider now the following nonlinear and nonautonomous periodic boundary value problem:

$$\begin{cases} u''(t) + a(t)u(t) = f(t, u(t)), & t \in I, \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \quad (5.3.1)$$

We will assume that the Hill's operator coupled with periodic conditions is non-resonant and $\lambda_0^A < 0$. From Lemma 5.2.1, we know that in this case the related Green's function changes its sign on $I \times I$.

On the other hand (as we have seen in Section 3.1), there exists v_P , a positive eigenfunction on I , unique up to a constant, related to λ_0^P , that is, v_P is such that

$$\begin{cases} v_P''(t) + a(t) v_P(t) = -\lambda_0^P v_P(t), & \text{for a. e. } t \in I, \\ v_P(0) = v_P(T), \quad v_P'(0) = v_P'(T). \end{cases}$$

Therefore,

$$v_P(t) = -\lambda_0^P \int_0^T G_P(t, s) v_P(s) \, ds$$

and, since v_P is positive and $\lambda_0^P < \lambda_0^A < 0$, we have that

$$\int_0^T G_P(t, s) v_P(s) \, ds > 0 \quad \forall t \in I$$

and, consequently,

$$\int_0^T G_P^+(t, s) v_P(s) \, ds > \int_0^T G_P^-(t, s) v_P(s) \, ds \quad \forall t \in I,$$

where G_P^+ and G_P^- are the positive and negative parts of G_P .

Since the Green's function changes sign, it makes sense to define

$$\gamma = \inf_{t \in I} \frac{\int_0^T G_P^+(t, s) v_P(s) \, ds}{\int_0^T G_P^-(t, s) v_P(s) \, ds} (> 1). \quad (5.3.2)$$

Moreover, in order to ensure the existence of solutions of problem (5.3.1), we will make the following assumptions:

(H_1) $f: I \times [0, \infty) \rightarrow [0, \infty)$ satisfies L^1 -Carathéodory conditions, that is:

- $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a. e. $t \in I$.
- For each $r > 0$ there exists $\phi_r \in L^1(I)$ such that

$$f(t, u) \leq \phi_r(t) \quad \text{for all } u \in [-r, r], \quad \text{a. e. } t \in I.$$

(H_2) There exist two positive constants m and M such that

$$m v_P(t) \leq f(t, x) \leq M v_P(t)$$

for every $t \in I$ and $x \geq 0$. Moreover, these constants satisfy that $\frac{M}{m} \leq \gamma$.

(H₃) There exists $[c, d] \subset I$ such that $\int_c^d G_P(t, s) \, dt \geq 0$, for all $s \in I$ and $\int_c^d G_P(t, s) \, dt > 0$, for all $s \in [c, d]$.

Remark 5.3.1. Note that condition (H₁) is the same as (L₀) given in Chapter 4 but, for the reader's convenience, we have decided to rewrite it so that one can find all the hypotheses used in this chapter together.

Remark 5.3.2. We note that condition (H₂) includes, as particular cases, hypotheses (J2) and (J3) in Theorem 5.1.1 used in [163]. This is due to the fact that, if $a(t) = \rho^2$, as in problem (5.1.4), we have that $\lambda_0^P = -\rho^2$ and $v_P(t) = 1$ for all $t \in I$. Moreover, as we will point out in Section 5.4, we have that, if $a(t) = \rho^2$, then

$$\int_0^T G_P(t, s) \, ds = \frac{1}{\rho^2},$$

and condition (H₃) is trivially fulfilled for $[c, d] = I$.

Moreover, we note that in (H₂) we are not considering the possibility of $m = 0$. Theorem 5.1.1 includes this case, but only when $\delta = +\infty$, which only happens when the Green's function is nonnegative. In [163] the authors consider this possibility because they are assuming that $\rho \in (0, \frac{3\pi}{2T}]$ and, when $\rho \in (0, \frac{\pi}{T}]$, G_P is nonnegative. As we will see in Corollary 5.3.5, hypothesis (H₂) is not necessary in case that the Green's function is nonnegative, so this is the reason why we do not consider the possibility $m = 0$.

We will consider the Banach space $(\mathcal{C}(I, \mathbb{R}), \|\cdot\|)$ coupled with the supremum norm $\|u\| \equiv \|u\|_\infty$, and define the cone

$$K = \left\{ u \in \mathcal{C}(I, \mathbb{R}) : u \geq 0 \text{ on } I, \int_0^T u(s) \, ds \geq \sigma \|u\| \right\},$$

where

$$\sigma = \frac{\eta}{\max_{t, s \in I} \{G_P(t, s)\}},$$

being

$$\eta = \min_{s \in [c, d]} \left\{ \int_c^d G_P(t, s) \, dt \right\} > 0. \quad (5.3.3)$$

Now, it is clear that u is a solution of the periodic problem (5.3.1) if and only if it is a fixed point of the following operator:

$$\mathcal{T}u(t) = \int_0^T G_P(t, s) f(s, u(s)) \, ds.$$

Lemma 5.3.3. Assume that $\lambda_0^A < 0$ and (H_1) – (H_3) hold. Then $\mathcal{T} : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is a compact operator which maps the cone K to itself.

Proof. We will divide the proof into several steps. We note that Steps 1 to 3 follow standard techniques but we include them for the sake of completeness.

Step 1: $\mathcal{T} : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is well-defined:

Let $u \in \mathcal{C}(I)$ and $(t_n)_{n \in \mathbb{N}} \subset I$ such that $\lim_{n \rightarrow \infty} t_n = t_0 \in I$.

On the one hand, from property $(G2)$ in the definition of Green's function (Definition 1.1.1), $G_P(\cdot, s)$ is uniformly continuous on I . Thus,

$$\lim_{n \rightarrow \infty} G_P(t_n, s) f(s, u(s)) = G_P(t_0, s) f(s, u(s)), \quad \text{a.e. } s \in I.$$

On the other hand it holds that

$$|G_P(t, s) f(s, u(s))| \leq |G_P(t, s)| \phi_{\|u\|}(s), \quad \text{a.e. } s \in I.$$

Moreover, from $(G2)$ in Definition 1.1.1, G_P is continuous on $I \times I$ and so it is bounded on $I \times I$ by some constant C . Therefore

$$|G_P(t, s) f(s, u(s))| \leq C \phi_{\|u\|}(s), \quad \text{a.e. } s \in I,$$

and, since the right hand side of previous inequality is in $L^1(I)$, by Lebesgue's Dominated Convergence Theorem we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{T}u(t_n) &= \lim_{n \rightarrow \infty} \int_0^T G_P(t_n, s) f(s, u(s)) \, ds \\ &= \int_0^T \lim_{n \rightarrow \infty} G_P(t_n, s) f(s, u(s)) \, ds \\ &= \int_0^T G_P(t_0, s) f(s, u(s)) \, ds = \mathcal{T}u(t_0). \end{aligned}$$

Thus, $\mathcal{T}u \in \mathcal{C}(I)$.

Step 2: Operator \mathcal{T} is continuous:

Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(I)$ be a sequence which converges to u in $\mathcal{C}(I)$. Then, there exists some $R \in \mathbb{R}^+$ such that $\|u_n\| \leq R$ for all $n \in \mathbb{N}$.

Now, from (H_1) , we deduce that

$$\lim_{n \rightarrow \infty} f(s, u_n(s)) = f(s, u(s)), \quad \text{for a.e. } s \in I.$$

On the other hand,

$$|G_P(t, s) f(s, u_n(s))| \leq C \phi_R(s), \quad \text{for a.e. } s \in I$$

and, since the right hand side of previous inequality is in $L^1(I)$, by Lebesgue's Dominated Convergence Theorem we deduce that

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathcal{T}u_n(t) &= \lim_{n \rightarrow \infty} \int_0^T G_P(t, s) f(s, u_n(s)) \, ds \\ &= \int_0^T \lim_{n \rightarrow \infty} G_P(t, s) f(s, u_n(s)) \, ds \\ &= \int_0^T G_P(t, s) f(s, u(s)) \, ds = \mathcal{T}u(t).\end{aligned}$$

Thus we can conclude that operator \mathcal{T} is continuous.

Step 3: \mathcal{T} is a compact operator:

Take

$$B = \{u \in \mathcal{C}(I) : \|u\| < r\}.$$

First, we will prove that $\mathcal{T}(B)$ is uniformly bounded:

$$\|\mathcal{T}u\| = \sup_{t \in I} \left| \int_0^T G_P(t, s) f(s, u(s)) \, ds \right| \leq \int_0^T C \phi_r(s) \, ds$$

and, since $\phi_r \in L^1(I)$, it is clear that $\mathcal{T}(B)$ is uniformly bounded.

Now, we will prove that \mathcal{T} is equicontinuous. We have that

$$\begin{aligned}|(\mathcal{T}u)(t_1) - (\mathcal{T}u)(t_2)| &\leq \int_0^T |G_P(t_1, s) - G_P(t_2, s)| f(s, u(s)) \, ds \\ &\leq \int_0^T |G_P(t_1, s) - G_P(t_2, s)| \phi_r(s) \, ds\end{aligned}$$

and, since G_P is uniformly continuous on $I \times I$, it occurs that for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|t_1 - t_2| < \delta$,

$$|(\mathcal{T}u)(t_1) - (\mathcal{T}u)(t_2)| \leq \varepsilon \int_0^T \phi_r(s) \, ds.$$

Thus, the fact that $\phi_r \in L^1(I)$ lets us conclude that \mathcal{T} is equicontinuous.

As a consequence, by Ascoli-Arzelà's Theorem (Theorem 1.2.2), we deduce that $\mathcal{T}(B)$ is relatively compact in $\mathcal{C}(I)$ and thus \mathcal{T} is a compact operator.

Step 4: \mathcal{T} maps the cone to itself.

Considering $u \in K$, then, from (5.3.2), the following inequalities are fulfilled for all $t \in I$:

$$\begin{aligned} \mathcal{T}u(t) &= \int_0^T G_P(t, s) f(s, u(s)) \, ds = \int_0^T (G_P^+(t, s) - G_P^-(t, s)) f(s, u(s)) \, ds \\ &\geq \int_0^T (m v_P(s) G_P^+(t, s) - M v_P(s) G_P^-(t, s)) \, ds \\ &\geq m \left(\int_0^T G_P^+(t, s) v_P(s) \, ds - \gamma \int_0^T G_P^-(t, s) v_P(s) \, ds \right) \geq 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^T \mathcal{T}u(t) \, dt &\geq \int_c^d \mathcal{T}u(t) \, dt = \int_c^d \int_0^T G_P(t, s) f(s, u(s)) \, ds \, dt \\ &= \int_0^T f(s, u(s)) \int_c^d G_P(t, s) \, dt \, ds \geq \eta \int_0^T f(s, u(s)) \, ds, \end{aligned}$$

and, since

$$\mathcal{T}u(t) \leq \max_{t, s \in I} \{G_P(t, s)\} \int_0^T f(s, u(s)) \, ds,$$

we deduce that

$$\int_0^T \mathcal{T}u(t) \, dt \geq \sigma \mathcal{T}u(t) \quad \text{for all } t \in I.$$

Thus,

$$\int_0^T \mathcal{T}u(t) \, dt \geq \sigma \|\mathcal{T}u\|,$$

and the result is concluded. \square

Now, in order to prove the existence of solutions of problem (5.3.1), we will use some classical results regarding the fixed point index which have been compiled in Lemma 1.2.7.

First, we note that, as an immediate consequence of condition (H_2) , we deduce the following properties:

$$f_0 = \lim_{x \rightarrow 0^+} \left\{ \min_{t \in [c, d]} \frac{f(t, x)}{x} \right\} = \infty, \quad f^\infty = \lim_{x \rightarrow \infty} \left\{ \max_{t \in I} \frac{f(t, x)}{x} \right\} = 0,$$

where the interval $[c, d]$ is given in (H_3) .

These properties will let us prove the following theorem.

Theorem 5.3.4. Assume that $\lambda_0^A < 0$ and hypotheses (H_1) – (H_3) hold. Then there exists at least one positive solution of problem (5.3.1) in the cone K .

Proof. Taking into account the definition of f_0 , we know that there exists $\delta_1 > 0$ such that when $\|u\| \leq \delta_1$, then

$$f(t, u(t)) > \frac{u(t)}{\eta}, \quad \forall t \in [c, d],$$

with η defined in (5.3.3).

Let

$$\Omega_1 = \{u \in K : \|u\| < \delta_1\}$$

and choose $u \in \partial\Omega_1$ and $e \in K \setminus \{0\}$.

We will prove that $u \neq \mathcal{T}u + \lambda e$ for every $\lambda > 0$.

Assume, on the contrary, that there exists some $\lambda > 0$ such that $u = \mathcal{T}u + \lambda e$, that is,

$$u(t) = \mathcal{T}u(t) + \lambda e(t) \geq \mathcal{T}u(t) \quad \forall t \in I.$$

Then

$$\begin{aligned} \int_c^d u(t) \, dt &\geq \int_c^d \mathcal{T}u(t) \, dt = \int_c^d \int_0^T G_P(t, s) f(s, u(s)) \, ds \, dt \\ &= \int_0^T \left(\int_c^d G_P(t, s) \, dt \right) f(s, u(s)) \, ds \\ &\geq \int_c^d \left(\int_c^d G_P(t, s) \, dt \right) f(s, u(s)) \, ds > \int_c^d u(s) \, ds, \end{aligned}$$

which is a contradiction.

Therefore, we deduce from Lemma 1.2.7 that $i_K(T, \Omega_1) = 0$.

Now, proceeding in an analogous way to [21, 62, 63], we define

$$\tilde{f}(t, u) = \max_{0 \leq z \leq u} f(t, z).$$

Clearly, $\tilde{f}(t, \cdot)$ is a nondecreasing function on $[0, \infty)$ and $\tilde{f}(t, x) \geq f(t, x)$ for all $t \in I, x \in [0, \infty)$. Moreover, since $f^\infty = 0$ it is obvious that

$$\lim_{x \rightarrow \infty} \left\{ \max_{t \in I} \frac{\tilde{f}(t, x)}{x} \right\} = 0.$$

As a consequence, we know that there exists $\delta_2 > 0$ such that if $\|u\| \geq \delta_2$ then

$$\tilde{f}(t, \|u\|) < \frac{\sigma^2}{T^2 \eta} \|u\|, \quad \forall t \in I.$$

Let

$$\Omega_2 = \{u \in K : \|u\| < \delta_2\}$$

and choose $u \in \partial \Omega_2$.

We will prove that $u \neq \mu \mathcal{T}u$ for every $\mu \leq 1$. Assume, on the contrary, that there exists some $\mu \leq 1$ such that $u(t) = \mu \mathcal{T}u(t)$ for all $t \in I$. Then,

$$\begin{aligned} \sigma \|u\| &\leq \int_0^T u(t) \, dt = \mu \int_0^T \mathcal{T}u(t) \, dt \\ &= \mu \int_0^T \int_0^T G_P(t, s) f(s, u(s)) \, ds \, dt \\ &= \mu \int_0^T \left(\int_0^T G_P(t, s) \, dt \right) f(s, u(s)) \, ds \\ &\leq \mu T \max_{t, s \in I} \{G_P(t, s)\} \int_0^T f(s, u(s)) \, ds \\ &\leq \mu T \max_{t, s \in I} \{G_P(t, s)\} \int_0^T \tilde{f}(s, u(s)) \, ds \\ &\leq \mu T \max_{t, s \in I} \{G_P(t, s)\} \int_0^T \tilde{f}(s, \|u\|) \, ds \\ &< \mu T^2 \frac{\eta}{\sigma} \frac{\sigma^2}{T^2 \eta} \|u\| \leq \sigma \|u\|, \end{aligned}$$

which is a contradiction. As a consequence, using Lemma 1.2.7, $i_K(\mathcal{T}, \Omega_2) = 1$.

We conclude, from Assertions 3 and 4 in Lemma 1.2.7, that the operator \mathcal{T} has a fixed point, that is, there exists at least a nontrivial solution of problem (5.3.1). \square

The previous theorem is also valid if the Green's function is nonnegative. In this case, hypothesis (H_3) would be trivially fulfilled and hypothesis (H_2) is not necessary since it is only used to prove that \mathcal{T} maps the cone to itself, which is obvious (since f is nonnegative) when G_P is nonnegative. On the other hand, we would need to add the hypothesis that $f_0 = \infty$ and $f^\infty = 0$ (which can not be deduced if we eliminate (H_2)).

The result is the following one:

Corollary 5.3.5. *Assume that $\lambda_0^P < 0 \leq \lambda_0^A$ and hypotheses (H_0) and (H_1) are fulfilled. Then, if $f_0 = \infty$ and $f^\infty = 0$ there exists at least one positive solution of problem (5.3.1) in the cone K .*

Remark 5.3.6. We note that for a nonnegative Green's function, we generalize the results of Graef, Kong and Wang [62, 63] and Webb [147] since our condition (H_3) is weaker than condition (5.1.1) considered by them.

Corollary 5.3.7. If $f(t, x) \equiv f(t)$, with $f \in L^1(I)$ satisfying (H_2) , then the unique solution of (5.3.1) is a nonnegative function on I .

Remark 5.3.8. We note that $u(t) \equiv 1$ is the unique solution of the periodic problem

$$\begin{cases} u''(t) + a(t)u(t) = a(t), & t \in I, \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases}$$

Therefore it is clear that

$$\int_0^T G_P(t, s) a(s) \, ds = 1 > 0 \quad (5.3.4)$$

and so the previous reasoning is also valid if $a \geq 0$, $a > 0$ on $[c, d]$, and we change the definition of γ by

$$\gamma^* = \inf_{t \in I} \frac{\int_0^T G_P^+(t, s) a(s) \, ds}{\int_0^T G_P^-(t, s) a(s) \, ds}.$$

In this case, assumption (H_2) would be substituted by

(H_2^*) There exist two positive constants m and M such that

$$m a(t) \leq f(t, x) \leq M a(t), \quad \text{for a.e. } t \in I, \quad x \geq 0.$$

Moreover, these constants satisfy that $\frac{M}{m} \leq \gamma^*$.

5.3.1. Neumann, Dirichlet and Mixed Boundary Value Problems

From the classical spectral theory (see Section 3.1), it is very well-known that, as in the periodic case, for any of the boundary conditions introduced in Lemma 5.2.1, there exists a positive eigenfunction on $(0, T)$ related to the correspondent smallest eigenvalue. Therefore, if we are in the case in which operator L coupled with the associated boundary conditions is nonresonant and the related Green's function changes sign (different cases are characterized in Lemma 5.2.1), we could follow the same argument as in the previous section to define γ , and we would obtain analogous existence results. Hypotheses (H_1) – (H_3) would be the same with the suitable notation for each of the problems (that is, considering in each case the appropriate Green's function and eigenfunction).

Remark 5.3.9. For the Neumann problem, it is not difficult to verify that we also have that if $a(t) = \rho^2$ then

$$\int_0^T G_N(t, s) \, ds = \frac{1}{\rho^2},$$

and condition (H_3) is trivially fulfilled for $[c, d] = I$.

On the other hand, since $u(t) \equiv 1$ is the unique solution of

$$\begin{cases} u''(t) + a(t)u(t) = a(t), & t \in I, \\ u'(0) = u'(T) = 0, \end{cases}$$

Remark 5.3.8 is also valid for the Neumann problem.

Remark 5.3.10. For the Dirichlet problem, condition (H_3) does not hold in case $[c, d] = I$. This is due to the fact that $G_D(t, \cdot)$ satisfies the Dirichlet boundary value conditions for all $t \in [0, T]$, that is, $G_D(t, 0) = G_D(t, T) = 0$.

It is important to note that the eigenfunction v_D is positive on $(0, T)$ but, since $v_D(0) = v_D(T) = 0$, condition (H_2) would imply that $f(0, x) = f(T, x) = 0$ for every $x \geq 0$. However, since as we have mentioned, $[c, d] \neq I$, this property does not affect on the fact that $f_0 = \infty$.

An analogous situation occurs for the mixed problems. In these cases it is also impossible to consider $[c, d] = I$ since the corresponding Green's functions and eigenfunctions vanish on one side of the interval.

Moreover, if we consider the Dirichlet and mixed problems, the constant function $u(t) \equiv 1$ is not a solution of the related linear problem $Lu(t) = a(t)$. Therefore, Remark 5.3.8 is not longer valid for such situations.

Remark 5.3.11. As it was commented in Remark 5.3.6, we also generalize the results of Graef, Kong and Wang [62, 63] and Webb [147] for a nonnegative Green's function coupled with the Neumann conditions.

On the other hand, as we have proved in Lemma 3.2.15, the Green's function G_D related to the second order Dirichlet problem

$$\begin{cases} u''(t) + a(t)u(t) = f(t, u(t)), & t \in I, \\ u(0) = u(T) = 0, \end{cases}$$

can never be nonnegative. However, if we consider the following problem

$$\begin{cases} -u''(t) - a(t)u(t) = f(t, u(t)), & t \in I, \\ u(0) = u(T) = 0, \end{cases} \quad (5.3.5)$$

it occurs that its related Green's function is $-G_D$. Therefore, our method would be applicable to prove the existence of constant sign solutions of problem (5.3.5) in case that $-G_D \geq 0$ on $I \times I$ (that is, when $G_D \leq 0$ on $I \times I$).

However, the results in [62, 63, 147] could not be applied to problem (5.3.5) since the related Green's function will cancel on the whole lines $s = 0$ and $s = T$ so the minimum in (5.1.1) would be 0. The same will happen with any mixed problem. Again, hypothesis (H_2) is not necessary in this case and we would need to add the hypothesis that $f_0 = \infty$ and $f^\infty = 0$.

5.4. Periodic Boundary Value Problem with Constant Potential

This section is devoted to the particular case in which the potential a is constant. As we will see, in this situation it is possible to calculate the exact value of γ .

It is easy to see that the eigenvalues associated to the periodic problem

$$\begin{cases} u'' + \lambda u = 0, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \quad (5.4.1)$$

are $\lambda_n^P = (2n\pi/T)^2$ with $n = 0, 1, 2, \dots$

Moreover, the eigenfunctions associated to the first eigenvalue $\lambda_0^P = 0$ are the constant functions, which can be written as multiples of a representative eigenfunction $v_P(t) \equiv 1$.

Furthermore, we know from Lemma 3.2.4, that the related Green's function is strictly negative in the square $I \times I$ if and only if $\lambda < \lambda_0^P = 0$ and it is nonnegative on $I \times I$ if and only if $0 = \lambda_0^P < \lambda \leq \lambda_0^A = (\pi/T)^2$.

In particular, for $\lambda = \rho^2 > \lambda_0^A$ a nonresonant value, the explicit expression of G_P is the following (see [16, 18, 104, 163]):

$$G_P(t, s) = \begin{cases} \frac{\sin \rho(t-s) + \sin \rho(T-t+s)}{2\rho(1 - \cos \rho T)}, & 0 \leq s \leq t \leq T, \\ \frac{\sin \rho(s-t) + \sin \rho(T-s+t)}{2\rho(1 - \cos \rho T)}, & 0 \leq t \leq s \leq T. \end{cases}$$

From (5.3.4), it is clear that

$$g(t) = \int_0^T G_P(t, s) \, ds = \frac{1}{\rho^2}.$$

Therefore, we define

$$\gamma = \min_{t \in [0, T]} \frac{\int_0^T G_P^+(t, s) \, ds}{\int_0^T G_P^-(t, s) \, ds} > 1,$$

for all $\rho > \pi/T$, $\rho \neq 2k\pi/T$, $k \in \mathbb{N}$.

Let us make a careful study of this value γ . From Lemma 2.4.3 and [18, Proposition 1.4.11], we know that the Green's function related to the periodic problem (5.4.1) satisfies that

$$G_P(t, s) = G_P(0, t - s) \text{ and } G_P(t, s) = G_P(T - t, T - s).$$

Therefore,

$$\int_0^T G_P(t, s) \, ds = \int_0^t G_P(t, s) \, ds + \int_t^T G_P(t, s) \, ds,$$

where

$$\int_0^t G_P(t, s) \, ds = \int_0^t G_P(0, t - s) \, ds = \int_0^t G_P(0, T + s - t) \, ds = \int_{T-t}^T G_P(0, s) \, ds$$

and

$$\int_t^T G_P(t, s) \, ds = \int_t^T G_P(0, T + s - t) \, ds = \int_T^{2T-t} G_P(0, s) \, ds = \int_0^{T-t} G_P(0, s) \, ds,$$

that is

$$\int_0^T G_P(t, s) \, ds = \int_0^T G_P(0, s) \, ds \text{ for all } t \in [0, T].$$

The same argument is valid for both the positive and the negative parts of G_P , that is

$$\int_0^T G_P^+(t, s) \, ds = \int_0^T G_P^+(0, s) \, ds \text{ and } \int_0^T G_P^-(t, s) \, ds = \int_0^T G_P^-(0, s) \, ds,$$

for all $t \in [0, T]$, so the ratio

$$\frac{\int_0^T G_P^+(t, s) \, ds}{\int_0^T G_P^-(t, s) \, ds}$$

is constant for all $t \in [0, T]$.

This implies that we can restrict our analysis to the case $t = 0$, that is, to assume that

$$\gamma = \frac{\int_0^T G_P^+(0, s) \, ds}{\int_0^T G_P^-(0, s) \, ds}.$$

We have that

$$G_P(0, s) = \frac{\sin \rho s + \sin \rho(T - s)}{2\rho(1 - \cos \rho T)},$$

so $G_P(0, s) = 0$ if and only if $s = \frac{T}{2} + \frac{(2k+1)\pi}{2\rho}$.

We will consider four cases:

- Case 1A: $G_P(0, \frac{T}{2}) G_P(0, 0) > 0$ and $G_P(0, \frac{T}{2}) > 0$.
- Case 1B: $G_P(0, \frac{T}{2}) G_P(0, 0) > 0$ and $G_P(0, \frac{T}{2}) < 0$.
- Case 2A: $G_P(0, \frac{T}{2}) G_P(0, 0) < 0$ and $G_P(0, \frac{T}{2}) > 0$.
- Case 2B: $G_P(0, \frac{T}{2}) G_P(0, 0) < 0$ and $G_P(0, \frac{T}{2}) < 0$.

Computing these values, we find that:

- If $\frac{(4k+1)\pi}{T} < \rho < \frac{(4k+2)\pi}{T}$ for some $k \in \mathbb{N}_0$, we are in Case 2A and

$$\gamma = \frac{2k+1}{2k+1 - \sin(\rho T/2)}.$$

- If $\frac{(4k+2)\pi}{T} < \rho < \frac{(4k+3)\pi}{T}$ for some $k \in \mathbb{N}_0$, we are in Case 2B and

$$\gamma = \frac{2k+1 - \sin(\rho T/2)}{2k+1}.$$

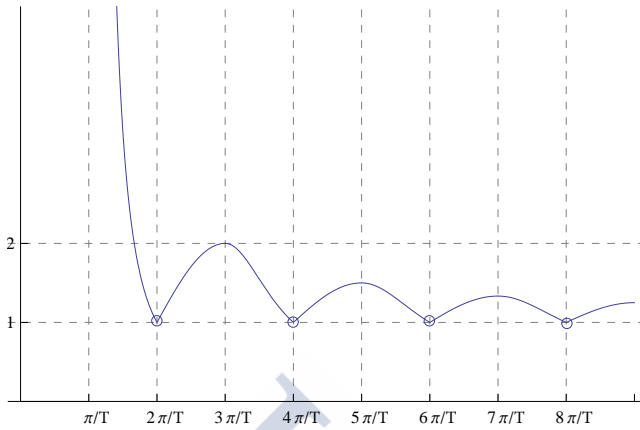
- If $\frac{(4k-1)\pi}{T} < \rho < \frac{4k\pi}{T}$ for some $k \in \mathbb{N}$, we are in Case 1B and

$$\gamma = \frac{2k}{2k + \sin(\rho T/2)}.$$

- If $\frac{4k\pi}{T} < \rho < \frac{(4k+1)\pi}{T}$ for some $k \in \mathbb{N}$, we are in Case 1A and

$$\gamma = \frac{2k + \sin(\rho T/2)}{2k}.$$

In the cases where $\rho = (2k+1)\frac{\pi}{T}$ for some $k \in \mathbb{N}$, the value of γ coincides with the limit when $\rho \rightarrow (2k+1)\frac{\pi}{T}$. The graph of γ for a given value ρ is sketched in Figure 5.4.1.


 Figure 5.4.1: Graph of γ for the periodic problem.

5.5. Dirichlet Boundary Value Problem with Constant Potential

Let us now try to calculate the value of γ for the Dirichlet problem with constant potential. In this case, the eigenvalues for the Dirichlet problem

$$\begin{cases} u''(t) + \lambda u(t) = 0, & \text{for } t \in I, \\ u(0) = u(T) = 0, \end{cases}$$

are $\lambda_n^D = ((n+1)\pi/T)^2$ for $n = 0, 1, 2, \dots$ and it follows easily that the eigenfunctions associated to $\lambda_0^D = (\pi/T)^2$ are the multiples of the function

$$v_D(t) = \sin\left(\frac{\pi t}{T}\right).$$

As we have proven in Lemma 3.2.15, the associated Green's function is strictly negative if and only if $\lambda < \lambda_0^D = (\pi/T)^2$, and it changes sign for any nonresonant value of $\lambda > (\pi/T)^2$.

Consider now $\lambda = \rho^2 > \lambda_0^D$ for $\rho \neq \frac{n\pi}{T}$, with $n \in \mathbb{N}$. We have that $v_D > 0$ is a solution of

$$\begin{cases} u''(t) + \lambda_0^D u(t) = 0, & t \in I, \\ u(0) = u(T) = 0, \end{cases}$$

and then it also solves

$$\begin{cases} u''(t) + \lambda u(t) = (\lambda - \lambda_0^D) v_D(t), & t \in I, \\ u(0) = u(T) = 0. \end{cases}$$

Therefore, since $\lambda = \rho^2 > \lambda_0^D$ is such that $\rho \neq \frac{n\pi}{T}$, with $n \in \mathbb{N}$, previous problem is nonresonant, and we deduce that

$$v_D(t) = \int_0^T (\lambda - \lambda_0^D) G_D(t, s) v_D(s) \, ds.$$

Since $v_D(t) > 0$ for $t \in (0, T)$, previous equality implies that

$$\int_0^T G_D(t, s) \sin\left(\frac{\pi s}{T}\right) \, ds > 0 \quad \text{for } t \in (0, T).$$

Thus, it makes sense to define

$$\gamma(\rho) = \inf_{t \in (0, T)} \gamma(t, \rho) = \inf_{t \in (0, T)} \frac{\int_0^T G_D^+(t, s) \sin\left(\frac{\pi s}{T}\right) \, ds}{\int_0^T G_D^-(t, s) \sin\left(\frac{\pi s}{T}\right) \, ds}.$$

The explicit formula for the Green's function in the nonresonant cases is given by (see [18])

$$G_D(t, s) = \begin{cases} G_1(t, s) = -\frac{\sin(\rho s) \sin \rho(T-t)}{\rho \sin(\rho T)}, & 0 \leq s \leq t \leq T, \\ G_2(t, s) = -\frac{\sin(\rho t) \sin \rho(T-s)}{\rho \sin(\rho T)}, & 0 \leq t \leq s \leq T. \end{cases} \quad (5.5.1)$$

We will consider two cases:

- **Case 1:** $\frac{(2n-1)\pi}{T} < \rho < \frac{2n\pi}{T}$ for $n \in \mathbb{N}$;
- **Case 2:** $\frac{2n\pi}{T} < \rho < \frac{(2n+1)\pi}{T}$ for $n \in \mathbb{N}$.

In Case 1 the function $\gamma(t, \rho)$ has a different computation in each of the $4n - 1$ intervals

$$\begin{aligned} & \left(0, T - \frac{(2n-1)\pi}{\rho T}\right], \left[T - \frac{(2n-1)\pi}{\rho T}, \frac{\pi}{\rho T}\right], \left[\frac{\pi}{\rho T}, T - \frac{(2n-2)\pi}{\rho T}\right], \\ & \left[T - \frac{(2n-2)\pi}{\rho T}, \frac{2\pi}{\rho T}\right], \dots, \left[\frac{(2n-2)\pi}{\rho T}, T - \frac{\pi}{\rho T}\right], \\ & \left[T - \frac{\pi}{\rho T}, \frac{(2n-1)\pi}{\rho T}\right], \left[\frac{(2n-1)\pi}{\rho T}, T\right) \end{aligned}$$

and in Case 2, it has a different computation in each of the $4n + 1$ intervals

$$\left(0, T - \frac{2n\pi}{\rho T}\right], \left[T - \frac{2n\pi}{\rho T}, \frac{\pi}{\rho T}\right], \dots, \left[T - \frac{\pi}{\rho T}, \frac{2n\pi}{\rho T}\right], \left[\frac{2n\pi}{\rho T}, T\right).$$

In both cases, given a fixed ρ it is easy to calculate the value of $\gamma(t, \rho)$. However the general expression for an arbitrary ρ requires very long computations which are not fundamental for the purpose of this chapter. Because of this, we are going to calculate the general expression of $\gamma(\rho)$ only for the first intervals of ρ , in particular for $\rho < \frac{6\pi}{T}$.

For $\rho < \frac{6\pi}{T}$, we can see that the infimum is attained at $t = 0$, so we will restrain our analysis to the first interval of t in both cases in order to obtain the exact expression of $\gamma(\rho)$ for $\rho < \frac{6\pi}{T}$.

In Case 1 we have

$$\begin{aligned} \int_0^T G_D^+(t, s) \sin\left(\frac{\pi s}{T}\right) ds &= \int_{T-\frac{\pi}{\rho}}^T G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds \\ &+ \sum_{i=2}^n \int_{T-\frac{(2i-1)\pi}{\rho T}}^{T-\frac{(2i-2)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds \end{aligned}$$

and

$$\begin{aligned} -\int_0^T G_D^-(t, s) \sin\left(\frac{\pi s}{T}\right) ds &= \int_0^t G_1(t, s) \sin\left(\frac{\pi s}{T}\right) ds \\ &+ \int_t^{T-\frac{(2n-1)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds \\ &+ \sum_{i=1}^{n-1} \int_{T-\frac{2i\pi}{\rho T}}^{T-\frac{(2i-1)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds \\ &= \frac{\sin\left(\frac{\pi t}{T}\right)}{\rho^2 - \left(\frac{\pi}{T}\right)^2} - \int_0^T G_D^+(t, s) \sin\left(\frac{\pi s}{T}\right) ds \end{aligned}$$

so

$$\gamma(t, \rho) = \frac{\int_{T-\frac{\pi}{\rho T}}^T G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds + \sum_{i=2}^n \int_{T-\frac{(2i-1)\pi}{\rho T}}^{T-\frac{(2i-2)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds}{\int_{T-\frac{\pi}{\rho T}}^T G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds + \sum_{i=2}^n \int_{T-\frac{(2i-1)\pi}{\rho T}}^{T-\frac{(2i-2)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds - \frac{\sin\left(\frac{\pi t}{T}\right)}{\rho^2 - \left(\frac{\pi}{T}\right)^2}}.$$

Doing a similar study for Case 2 we get

$$\gamma(t, \rho) = \frac{\sum_{i=1}^n \int_{T-\frac{2i\pi}{\rho T}}^{T-\frac{(2i-1)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds}{\sum_{i=1}^n \int_{T-\frac{2i\pi}{\rho T}}^{T-\frac{(2i-1)\pi}{\rho T}} G_2(t, s) \sin\left(\frac{\pi s}{T}\right) ds - \frac{\sin\left(\frac{\pi t}{T}\right)^2}{\rho^2 - \left(\frac{\pi}{T}\right)^2}}.$$

Using the previous expressions, it is immediate to calculate $\gamma(t, \rho)$ for any fixed value of ρ and T . For instance, computing $\gamma(t, \rho)$ for $T = 1$ we obtain:

- If $\rho \in (\pi, 2\pi)$, then

$$\gamma(t, \rho) = \frac{\sin \rho t \sin \frac{\pi^2}{\rho}}{\sin \rho t \sin \frac{\pi^2}{\rho} + \sin \rho \sin \pi t}.$$

- If $\rho \in (2\pi, 3\pi)$, then

$$\gamma(t, \rho) = \frac{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} \right)}{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} \right) - \sin \rho \sin \pi t}.$$

- If $\rho \in (3\pi, 4\pi)$, then

$$\gamma(t, \rho) = \frac{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} \right)}{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} \right) + \sin \rho \sin \pi t}.$$

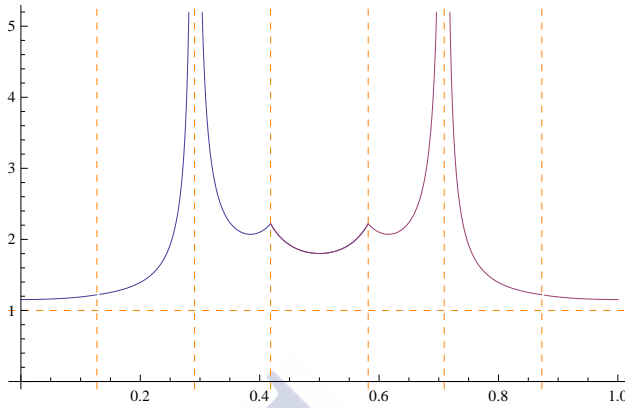
- If $\rho \in (4\pi, 5\pi)$, then

$$\gamma(t, \rho) = \frac{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} \right)}{\sin \rho t \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} \right) - \sin \rho \sin \pi t}.$$

- If $\rho \in (5\pi, 6\pi)$, then

$$\gamma(t, \rho) = \frac{\sin \rho t \left(\sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} + \sin \frac{5\pi^2}{\rho} \right) + 2 \left(1 - \frac{\pi^2}{\rho^2} \right) \sin \rho t}{\sin \rho t \left(\sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} + \sin \frac{5\pi^2}{\rho} \right) + \sin \rho \sin \pi t + 2 \left(1 - \frac{\pi^2}{\rho^2} \right) \sin \rho t}.$$

In Figure 5.5.1 we have a sketch of the function $\gamma(t, 10.8)$ for $T = 1$.


 Figure 5.5.1: Graph of $\gamma(t, 10.8)$ for the Dirichlet problem.

Computing the limit

$$\gamma(\rho) = \lim_{t \rightarrow 0} \gamma(t, \rho),$$

we get the following expressions for $\gamma(\rho)$:

- If $\rho \in (\pi, 2\pi)$, then

$$\gamma(\rho) = 1 - \frac{\pi \sin \rho}{\pi \sin \rho + \rho \sin \frac{\pi^2}{\rho}}.$$

- If $\rho \in (2\pi, 3\pi)$, then

$$\gamma(\rho) = 1 + \frac{\pi \sin \rho}{-\pi \sin \rho + \rho \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} \right)}.$$

- If $\rho \in (3\pi, 4\pi)$, then

$$\gamma(\rho) = 1 - \frac{\pi \sin \rho}{\pi \sin \rho + \rho \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} \right)}.$$

- If $\rho \in (4\pi, 5\pi)$, then

$$\gamma(\rho) = 1 + \frac{\pi \sin \rho}{-\pi \sin \rho + \rho \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} \right)}.$$

- If $\rho \in (5\pi, 6\pi)$, then

$$\gamma(\rho) = 1 - \frac{\pi \sin \rho}{\pi \sin \rho + \rho \left(\sin \frac{\pi^2}{\rho} + \sin \frac{2\pi^2}{\rho} + \sin \frac{3\pi^2}{\rho} + \sin \frac{4\pi^2}{\rho} + \sin \frac{5\pi^2}{\rho} \right) + 2 \frac{\rho^2 - \pi^2}{\rho}}.$$

Graphically the function $\gamma(\rho)$ is represented in Figure 5.5.2 for $T = 1$.

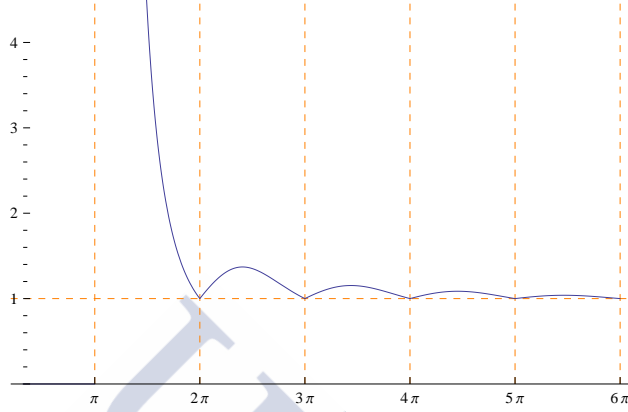


Figure 5.5.2: Graph of γ for the Dirichlet problem.

Remark 5.5.1. Analogous arguments and calculations can be done for Neumann and mixed problems.

Let us now see an example.

Example 5.5.2. Consider the Dirichlet problem

$$\begin{cases} u''(t) + 60u(t) = t(1-t), & t \in [0, 1], \\ u(0) = u(1) = 0. \end{cases} \quad (5.5.2)$$

It holds that $\gamma(\sqrt{60}) \approx 1.36 > \frac{4}{3}$ and

$$\frac{3}{4} \frac{\sin(\pi t)}{\pi} \leq t(1-t) \leq \frac{\sin(\pi t)}{\pi},$$

so hypothesis (H_2) is satisfied for $m = \frac{3}{4}$ and $M = 1$.

Thus, from Corollary 5.3.7, the unique solution of problem (5.5.2) is nonnegative on $[0, 1]$ (in particular, it is positive on $(0, 1)$).

However, the solution of the Dirichlet problem

$$\begin{cases} u''(t) + 60u(t) = t, & t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \quad (5.5.3)$$

changes sign.

We can see the respective solutions in Figures 5.5.3 and 5.5.4.

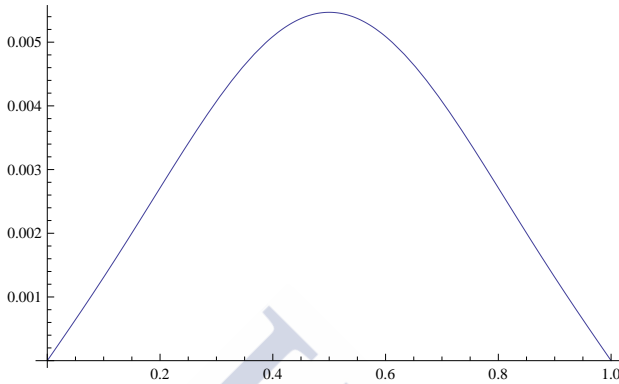


Figure 5.5.3: Solution of problem (5.5.2).

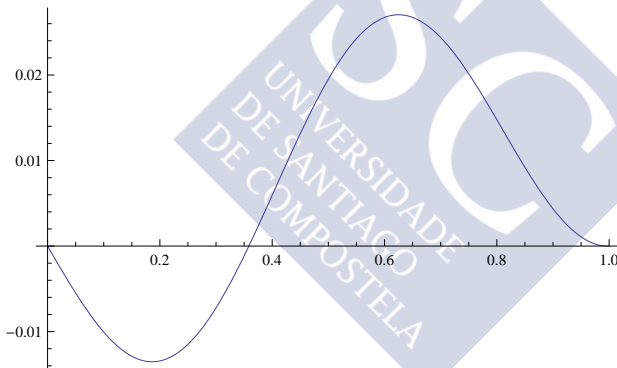


Figure 5.5.4: Solution of problem (5.5.3).

We will finish this chapter with an example of a nonlinear problem.

Example 5.5.3. Consider now the following nonlinear Dirichlet problem

$$\begin{cases} u''(t) + 60u(t) = t(1-t) \frac{3 + 29(u(t))^2}{1 + 10(u(t))^2}, & t \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \quad (5.5.4)$$

that is, we are considering

$$f(t, x) = t(1-t) \frac{3 + 29x^2}{1 + 10x^2}.$$

We will see that this problem satisfies conditions (H_1) – (H_2) :

(H_1) It is immediate to verify that this condition holds for $\phi_r(t) = t(1 - t)$.

(H_2) It can be checked that

$$2.26 \frac{\sin(\pi t)}{\pi} \leq t(1 - t) \frac{3 + 29x^2}{1 + 10x^2} \leq 3 \frac{\sin(\pi t)}{\pi}$$

for all $x \in \mathbb{R}$ and $t \in [0, 1]$.

Thus, since $\gamma(\sqrt{60}) \approx 1.36$ and

$$\frac{3}{2.26} \approx 1.33 < \gamma(\sqrt{60}),$$

hypothesis (H_2) holds for $m = 2.26$ and $M = 3$.

(H_3) Using [24] we obtain the explicit expression of the Green's function related to this problem

$$G_D(t, s) = \frac{-1}{2\sqrt{15}} \begin{cases} \frac{\sin 2\sqrt{15}(1-s) \sin 2\sqrt{15}t}{\sin 2\sqrt{15}} + \sin 2\sqrt{15}(t-s), & 0 \leq s \leq t \leq 1, \\ \frac{\sin 2\sqrt{15}(1-s) \sin 2\sqrt{15}t}{\sin 2\sqrt{15}}, & 0 \leq t \leq s \leq 1. \end{cases}$$

As we can see in Figure 5.5.5, this Green's function changes sign on its square of definition.

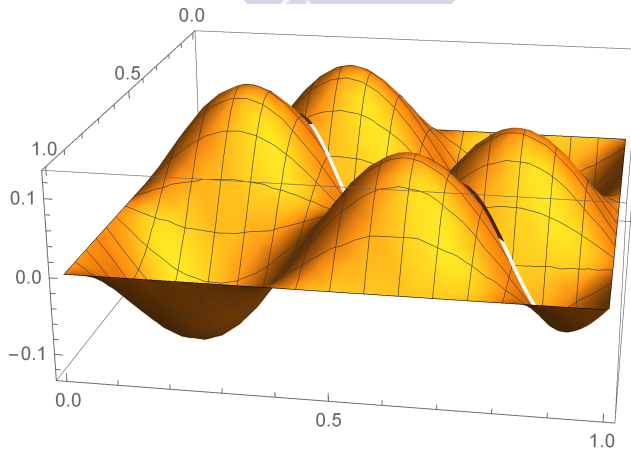


Figure 5.5.5: Green's function related to problem (5.5.4).

Finally, if we take $[c, d] = [\frac{1}{4}, \frac{3}{4}]$, it occurs that

$$\int_{\frac{1}{4}}^{\frac{3}{4}} G_D(t, s) \, dt \geq 0, \quad \text{for all } s \in [0, 1].$$

In particular,

$$\int_{\frac{1}{4}}^{\frac{3}{4}} G_D(t, s) \, dt > 0, \quad \text{for all } s \in (0, 1)$$

and

$$\int_{\frac{1}{4}}^{\frac{3}{4}} G_D(t, 0) \, dt = \int_{\frac{1}{4}}^{\frac{3}{4}} G_D(t, 1) \, dt = 0.$$

Thus, hypothesis (H_3) holds for $[c, d] = [\frac{1}{4}, \frac{3}{4}]$.

We note that, as we have mentioned in Remark 5.3.10, it is not possible to choose $[c, d] = [0, 1]$ in this case.

Moreover, by numerical approach, we obtain the following values for the constants involved in the construction of the cone

$$\eta \approx 0.0087 \quad \text{and} \quad \sigma \approx 0.067.$$

Since all the hypotheses hold, we can conclude that problem (5.5.4) has a nontrivial and nonnegative solution in the cone

$$K = \left\{ u \in \mathcal{C}([0, 1], \mathbb{R}) : u \geq 0 \text{ on } [0, 1], \int_0^1 u(s) \, ds \geq \sigma \|u\| \right\}.$$

Existence and Multiplicity Results for some Generalized Hammerstein Equations with a Parameter

As we have seen in previous chapters, every nonlinear differential equation can be transformed into an integral operator whose fixed points will correspond with the solutions of the aforementioned differential equation.

Therefore it is common to work directly with integral problems defined on Banach spaces, the so-called Hammerstein equations, which depend on a kernel function. These kernel functions include but are not limited to Green's functions. This way, the study of fixed points of Hammerstein integral equations is further more general than the one of nonlinear differential equations.

The solvability of this type of integral equations has been considered by many authors. In fact they have become both a generalization of differential equations and boundary value problems and a main field for applications of methods and techniques of nonlinear analysis, as it can be seen, for instance, in [7, 57, 69, 79–81, 83, 116].

Sections 6.1 to 6.6 in this chapter are included in [102], while the particular case showed in Section 6.7 is collected in [32].

6.1. Introduction

We will study the existence and multiplicity of fixed points of the integral operator

$$\mathcal{T}u(t) = \lambda \int_0^T k(t, s) f(s, u(s), u'(s), \dots, u^{(m)}(s)) \, ds, \quad t \in I, \quad (6.1.1)$$

where

- $\lambda > 0$ is a positive parameter,
- $k : I \times I \rightarrow \mathbb{R}$ is a kernel function such that $k \in W^{m,1}(I \times I)$,

- m is a positive integer ($m \geq 1$) and
- $f : I \times \mathbb{R}^{m+1} \rightarrow [0, +\infty)$ is a L^1 -Carathéodory function.

As it has been said before, this type of integral equations are known as Hammerstein equations (see [71]).

In [61], the following generalized Hammerstein equation is studied

$$u(t) = \int_0^1 k(t, s) g(s) f(s, u(s), u'(s), \dots, u^{(m)}(s)) \, ds, \quad (6.1.2)$$

with $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ a kernel function such that $k \in W^{m,1}([0, 1] \times [0, 1])$, $m \geq 1$ is an integer, $g \in L^1([0, 1])$ is nonnegative almost everywhere in $[0, 1]$, and $f : [0, 1] \times \mathbb{R}^{m+1} \rightarrow [0, \infty)$ a L^∞ -Carathéodory function. Moreover, both the kernel $k(t, s)$ and its derivatives $\frac{\partial^i k}{\partial t^i}(t, s)$, for $i = 1, \dots, m$, are bounded and nonnegative on the square $[0, 1] \times [0, 1]$.

Our work generalize the existing results in the literature introducing a new type of cone,

$$K = \left\{ u \in C^m(I, \mathbb{R}) : \begin{array}{l} u^{(i)}(t) \geq 0, \, t \in [m_i, n_i], \, i \in J_0; \\ \min_{t \in [a_j, b_j]} u^{(j)}(t) \geq \xi_j \|u^{(j)}\|_{[c_j, d_j]}, \, j \in J_1 \end{array} \right\},$$

where

$$\|u^{(j)}\|_{[c_j, d_j]} := \max_{t \in [c_j, d_j]} |u^{(j)}(t)|,$$

$J \equiv \{0, 1, \dots, m\}$ and $J_1 \subset J_0 \subset J$, $J_1 \neq \emptyset$.

We note that the nonnegativeness of the functions and their derivatives may happen only on a subinterval, possibly degenerate (that is, reduced to a point), and, as $J_1 \subset J$, $J_1 \neq \emptyset$, the second property can hold, locally, only for a restrict number of derivatives, including the function itself. This way, it is not required, as it was usual, that $k(t, s)$ and $\frac{\partial^i k}{\partial t^i}(t, s)$ have constant sign on the square of definition.

Another important novelty is that, in the second property of the cone, we are considering the norm of the functions on a subset of the domain and not on the whole interval $[0, T]$. Moreover, as we will see, the two subintervals involved in this condition, $[a_j, b_j]$ and $[c_j, d_j]$, must have nonempty intersection but are not required to satisfy any other inclusion property (that is, it may occur that $[a_j, b_j] \not\subset [c_j, d_j]$ and $[c_j, d_j] \not\subset [a_j, b_j]$).

In particular, second property in the cone (which ensures that the minimum of the function and the derivatives on some interval is bigger than its norm on another interval) will be given by certain inequalities (introduced in (H_4)) that both the kernel

and its derivatives must hold. However, contrary to recent references such as [61], the bounds must hold only for, at least, one of the derivatives of the kernel or, even, for the kernel, on a subset of the domain.

This chapter is organized in the following way: Section 6.2 contains the main assumptions, the definition of the new cone and some properties on the integral operator. In Section 6.3, the existence results are obtained with several asymptotic assumptions on f of the sublinear or superlinear type, near 0 or $+\infty$. Section 6.4 presents existence and multiplicity results applying fixed point index theory. Section 6.5 has two examples to illustrate our main results and, moreover, to emphasize the importance that (H_4) holds only for some derivatives and that the subsets could be reduced to a point. Section 6.6 contains an application to $2n$ -th order Dirichlet problems (also called Lidstone problems), giving new sufficient conditions for the solvability of these problems, which allow the dependence of the nonlinearity on odd and even derivatives. In fact, our method allows that the nonlinearities may depend on derivatives of even and odd order, which is new in the literature on this type of problems, as it can be seen, for instance, in [43, 86, 146, 159]. In this way, our results fill some gaps and improve the study of Lidstone and complementary Lidstone problems. Finally, Section 6.7 shows a particular case of a third order three-point boundary value problem which is solved using the results developed in this chapter. This last section gives also some conditions under which the considered problem has not any nontrivial solution.

6.2. Hypotheses and Auxiliary Results

Let's consider $E = C^m(I, \mathbb{R})$ equipped with the norm

$$\|u\| = \max\{\|u^{(i)}\|_\infty, i \in J\},$$

where $\|v\|_\infty = \sup_{t \in I} |v(t)|$.

It is very well-known that $(E, \|\cdot\|)$ is a Banach space.

Throughout this chapter we will make the following assumptions:

- (H_1) The kernel function $k: I \times I \rightarrow \mathbb{R}$ is such that $k \in W^{m,1}(I \times I)$, with $m \geq 1$. Moreover, for $i = 0, \dots, m-1$, it holds that for every $\varepsilon > 0$ and every fixed $\tau \in I$, there exists some $\delta > 0$ such that $|t - \tau| < \delta$ implies that

$$\left| \frac{\partial^i k}{\partial t^i}(t, s) - \frac{\partial^i k}{\partial t^i}(\tau, s) \right| < \varepsilon \text{ for a.e. } s \in I.$$

Finally, for the m -th derivative of the kernel, it holds that, for every $\varepsilon > 0$ and every fixed $\tau \in I$, there exist a set $Z_\tau \in I$ with measure equal to zero and some

$\delta > 0$ such that $|t - \tau| < \delta$ implies that

$$\left| \frac{\partial^m k}{\partial t^m}(t, s) - \frac{\partial^m k}{\partial t^m}(\tau, s) \right| < \varepsilon,$$

for all $s \in I \setminus Z_\tau$ such that $s < \min\{t, \tau\}$ or $s > \max\{t, \tau\}$.

(H₂) For each $i \in J_0 \subset J$, $J_0 \neq \emptyset$, there exists a subinterval $[m_i, n_i]$ such that

$$\frac{\partial^i k}{\partial t^i}(t, s) \geq 0 \text{ for all } t \in [m_i, n_i], s \in I.$$

It is possible that this interval is degenerated, that is, $m_i = n_i$.

(H₃) For all $i \in J$, there exist positive functions $h_i \in L^1(I)$ such that

$$\left| \frac{\partial^i k}{\partial t^i}(t, s) \right| \leq h_i(s) \text{ for all } t \in I \text{ and a.e. } s \in I.$$

(H₄) For each $j \in J_1 \subset J_0$, $J_1 \neq \emptyset$, there exist subintervals $[a_j, b_j] \subset [m_j, n_j]$ and $[c_j, d_j]$, with $[a_j, b_j] \cap [c_j, d_j] \neq \emptyset$, positive functions $\phi_j: I \rightarrow [0, \infty)$ and constants $\xi_j \in (0, 1)$ such that

$$\left| \frac{\partial^j k}{\partial t^j}(t, s) \right| \leq \phi_j(s) \text{ for all } t \in [c_j, d_j] \text{ and a.e. } s \in I,$$

and

$$\frac{\partial^j k}{\partial t^j}(t, s) \geq \xi_j \phi_j(s) \text{ for all } t \in [a_j, b_j] \text{ and a.e. } s \in I.$$

Moreover, $\phi_j \in L^1(I)$ satisfies that

$$\int_{a_j}^{b_j} \phi_j(s) \, ds > 0.$$

(H₅) There exists $i_0 \in J_0$ such that either $[c_{i_0}, d_{i_0}] \equiv I$ or $[m_{i_0}, n_{i_0}] \equiv I$ and, moreover, $\{0, 1, \dots, i_0\} \subset J_0$.

(H₆) The nonlinearity $f: I \times \mathbb{R}^{m+1} \rightarrow [0, \infty)$ satisfies L^1 -Carathéodory conditions, that is,

- $f(\cdot, x_0, \dots, x_m)$ is measurable for each (x_0, \dots, x_m) fixed.
- $f(t, \cdot, \dots, \cdot)$ is continuous for a.e. $t \in I$.

- For each $r > 0$ there exists $\varphi_r \in L^1(I)$ such that

$$f(t, x_0, \dots, x_m) \leq \varphi_r(t), \quad \forall (x_0, \dots, x_m) \in (-r, r)^{m+1}, \quad \text{a.e. } t \in I.$$

(H_7) Functions h_i defined in (H_3) and φ_r defined in (H_6) satisfy that $h_i \varphi_r \in L^1(I)$ for every $i \in J$ and $r > 0$.

We will look for fixed points of operator \mathcal{T} on a suitable cone on the Banach space E .

In particular, taking into account the properties satisfied by the kernel k , we define the cone

$$K = \left\{ u \in C^m(I, \mathbb{R}) : \begin{array}{l} u^{(i)}(t) \geq 0, \quad t \in [m_i, n_i], \quad i \in J_0; \\ \min_{t \in [a_j, b_j]} u^{(j)}(t) \geq \xi_j \|u^{(j)}\|_{[c_j, d_j]}, \quad j \in J_1 \end{array} \right\}.$$

Lemma 6.2.1. *Hypothesis (H_5) guarantees that K is a cone in E .*

Proof. We need to verify that K is a closed and convex subset of $C^m(I, \mathbb{R})$ and that satisfies the two properties which characterize cones in a Banach space (see Definition 1.2.4).

First of all, from the definition of K , it is clear that it is closed. We will see that it is convex. For $u, v \in K$ and $\lambda \in (0, 1)$, it is clear that

$$(1 - \lambda) u^{(i)}(t) + \lambda v^{(i)}(t) \geq 0, \quad \text{for } t \in [m_i, n_i], \quad i \in J_0.$$

In addition, for $j \in J_1$,

$$\begin{aligned} \min_{t \in [a_j, b_j]} \left((1 - \lambda) u^{(j)}(t) + \lambda v^{(j)}(t) \right) &\geq (1 - \lambda) \min_{t \in [a_j, b_j]} u^{(j)}(t) + \lambda \min_{t \in [a_j, b_j]} v^{(j)}(t) \\ &\geq (1 - \lambda) \xi_j \|u^{(j)}\|_{[c_j, d_j]} + \lambda \xi_j \|v^{(j)}\|_{[c_j, d_j]} \\ &= \xi_j \left((1 - \lambda) \|u^{(j)}\|_{[c_j, d_j]} + \lambda \|v^{(j)}\|_{[c_j, d_j]} \right) \\ &\geq \xi_j \|(1 - \lambda) u^{(j)} + \lambda v^{(j)}\|_{[c_j, d_j]}. \end{aligned}$$

Thus, $(1 - \lambda) u + \lambda v \in K$.

Moreover, from the definition of K , it is trivial to check that if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$.

Now, to prove that $K \cap (-K) = \{0\}$, we will distinguish between two different cases:

(I) There exists $i_0 \in J_0$ such that $[m_{i_0}, n_{i_0}] \equiv I$.

Suppose that $u, -u \in K$. Then $u^{(i_0)}(t) \geq 0$ and $-u^{(i_0)}(t) \geq 0$ for all $t \in I$, which implies that $u^{(i_0)} \equiv 0$ on I . If $i_0 \geq 1$, $u^{(i_0-1)}$ is constant on I .

Now, we have that $u^{(i_0-1)}(t) \geq 0$ and $-u^{(i_0-1)}(t) \geq 0$ for all $t \in [m_{i_0-1}, n_{i_0-1}]$, that is $u^{(i_0-1)} \equiv 0$ on $[m_{i_0-1}, n_{i_0-1}]$. Then, since $u^{(i_0-1)}$ is constant on I , we deduce that $u^{(i_0-1)} \equiv 0$ on I .

Using the same argument repeatedly, we conclude that $u \equiv 0$ on I . In this way, we have proved that $K \cap (-K) = \{0\}$.

(II) There exists $i_0 \in J_0$ such that $[c_{i_0}, d_{i_0}] \equiv I$.

Suppose again that $u, -u \in K$. Then, from the fact that

$$\min_{t \in [a_{i_0}, b_{i_0}]} u^{(i_0)}(t) \geq \xi_{i_0} \|u^{(i_0)}\|_I \quad \text{and} \quad \min_{t \in [a_{i_0}, b_{i_0}]} (-u^{(i_0)}(t)) \geq \xi_{i_0} \|u^{(i_0)}\|_I,$$

we deduce that $\|u^{(i_0)}\|_I = 0$, which implies that $u^{(i_0)} \equiv 0$ on I . Now, following the same arguments than in Case (I), we conclude the result. □

In the next section, considering some additional properties on the function f , we will ensure the existence of fixed points of operator \mathcal{T} . However, before doing that, we need to prove that \mathcal{T} is compact.

Lemma 6.2.2. *If hypotheses (H_1) – (H_7) hold, then operator $\mathcal{T}: K \rightarrow K$ defined in (6.1.1) is compact.*

Proof. We divide the proof into several steps.

Step 1. \mathcal{T} is well defined in K .

Let $u \in K$.

First we will prove that $\mathcal{T}u \in \mathcal{C}^m(I, \mathbb{R})$. By the rule of differentiation under the integral sign for Lebesgue's integral (see [15, 139, 140]), we have that for all $i \in J$

$$(\mathcal{T}u)^{(i)}(t) = \lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds.$$

Now let $(t_n)_{n \in \mathbb{N}} \subset I$ such that $\lim_{n \rightarrow \infty} t_n = t_0 \in I$. On the one hand, by (H_1) , we have that

$$\lim_{n \rightarrow \infty} \frac{\partial^i k}{\partial t^i}(t_n, s) f(s, u(s), \dots, u^{(m)}(s)) = \frac{\partial^i k}{\partial t^i}(t_0, s) f(s, u(s), \dots, u^{(m)}(s)),$$

for a. e. $s \in I$.

On the other hand,

$$\left| \frac{\partial^i k}{\partial t^i}(t_n, s) f(s, u(s), \dots, u^{(m)}(s)) \right| \leq h_i(s) \varphi_{\|u\|}(s)$$

and so, by Lebesgue's Dominated Convergence Theorem and (H_7) , we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} (Tu)^{(i)}(t_n) &= \lim_{n \rightarrow \infty} \lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t_n, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &= \lambda \int_0^T \lim_{n \rightarrow \infty} \frac{\partial^i k}{\partial t^i}(t_n, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &= (Tu)^{(i)}(t_0). \end{aligned}$$

Thus, $(Tu)^{(i)}$ is continuous on I for $i = 0, \dots, m$, that is, $Tu \in \mathcal{C}^m(I, \mathbb{R})$.

Now, we will prove that $Tu \in K$.

It is obvious that, for $i \in J_0$, $(Tu)^{(i)}(t) \geq 0$ for all $t \in [m_i, n_i]$.

Moreover, for $j \in J_1$ and $t \in [c_j, d_j]$, we have that

$$\begin{aligned} |(\mathcal{T}u)^{(j)}(t)| &\leq \lambda \int_0^T \left| \frac{\partial^j k}{\partial t^j}(t, s) \right| f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\leq \lambda \int_0^T \phi_j(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds, \end{aligned}$$

and, taking the supremum for $t \in [c_j, d_j]$, we deduce that

$$\|(\mathcal{T}u)^{(j)}\|_{[c_j, d_j]} \leq \lambda \int_0^T \phi_j(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds.$$

Moreover, for $t \in [a_j, b_j]$, we have

$$\begin{aligned} (\mathcal{T}u)^{(j)}(t) &= \lambda \int_0^T \frac{\partial^j k}{\partial t^j}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\geq \lambda \int_0^T \xi_j \phi_j(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \geq \xi_j \|(\mathcal{T}u)^{(j)}\|_{[c_j, d_j]} \end{aligned}$$

and we deduce that

$$\min_{t \in [a_j, b_j]} (\mathcal{T}u)^{(j)}(t) \geq \xi_j \|(\mathcal{T}u)^{(j)}\|_{[c_j, d_j]}$$

for $j \in J_1$.

Therefore, we can conclude that $\mathcal{T}u \in K$.

Step 2. \mathcal{T} is continuous in $\mathcal{C}^m(I, \mathbb{R})$.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence which converges to u in $\mathcal{C}^m(I, \mathbb{R})$. Then, $u_n^{(i)}(s)$ converges to $u^{(i)}(s)$ and, from (H_6) , this implies that $f(s, u_n(s), \dots, u_n^{(m)}(s))$ converges to $f(s, u(s), \dots, u^{(m)}(s))$ for a.e. $s \in I$. Multiplying by $h_i(s)$, it is clear that

$$\lim_{n \rightarrow \infty} h_i(s) f(s, u_n(s), \dots, u_n^{(m)}(s)) = h_i(s) f(s, u(s), \dots, u^{(m)}(s)) \text{ for a.e. } s \in I.$$

On the one hand, it is clear that there exists some $R \in \mathbb{R}^+$ for which $\|u_n\| \leq R$ for all $n \in \mathbb{N}$. Therefore,

$$\left| h_i(s) f(s, u_n(s), \dots, u_n^{(m)}(s)) \right| \leq h_i(s) \varphi_R(s), \quad \text{a.e. } s \in I.$$

Since, by (H_7) , $h_i \varphi_R \in L^1(I)$, by Lebesgue's Dominated Convergence Theorem we deduce that

$$\lim_{n \rightarrow \infty} \int_0^T h_i(s) \left| f(s, u_n(s), \dots, u_n^{(m)}(s)) - f(s, u(s), \dots, u^{(m)}(s)) \right| ds = 0.$$

On the other hand, for $t \in I$,

$$\begin{aligned} & \left| (\mathcal{T}u_n)^{(i)}(t) - (\mathcal{T}u)^{(i)}(t) \right| \\ & \leq \lambda \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| \left| f(s, u_n(s), \dots, u_n^{(m)}(s)) - f(s, u(s), \dots, u^{(m)}(s)) \right| ds \\ & \leq \lambda \int_0^T h_i(s) \left| f(s, u_n(s), \dots, u_n^{(m)}(s)) - f(s, u(s), \dots, u^{(m)}(s)) \right| ds \end{aligned}$$

and therefore

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| (\mathcal{T}u_n)^{(i)} - (\mathcal{T}u)^{(i)} \right\|_{\infty} \\ & = \lim_{n \rightarrow \infty} \left(\sup_{t \in I} \left| (\mathcal{T}u_n)^{(i)}(t) - (\mathcal{T}u)^{(i)}(t) \right| \right) \\ & \leq \lim_{n \rightarrow \infty} \lambda \int_0^T h_i(s) \left| f(s, u_n(s), \dots, u_n^{(m)}(s)) - f(s, u(s), \dots, u^{(m)}(s)) \right| ds = 0, \end{aligned}$$

from where we conclude the uniform convergence of $\mathcal{T}u_n$ to $\mathcal{T}u$ on I . Thus, operator \mathcal{T} is continuous.

Step 3. \mathcal{T} is a compact operator.

Let's consider

$$B = \{u \in E; \|u\| \leq r\}.$$

First, we will prove that $T(B)$ is uniformly bounded in $\mathcal{C}^m(I)$.

We find the following bounds for $u \in B$ and $i \in J$:

$$\begin{aligned} \|(\mathcal{T}u)^{(i)}\|_\infty &= \sup_{t \in I} \left| \lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \right| \\ &\leq \lambda \int_0^T h_i(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\leq \lambda \int_0^T h_i(s) \varphi_r(s) \, ds := M_i, \end{aligned}$$

with $M_i > 0$. Therefore,

$$\|\mathcal{T}u\| \leq \max\{M_i : i \in J\}, \quad \forall u \in B.$$

Now, we will prove that $\mathcal{T}(B)$ is equicontinuous in $\mathcal{C}^m(I)$. Let $t_2 \in I$ be fixed. Then, for every $\varepsilon > 0$, take $\delta > 0$ given in (H_1) and for $i = 0, \dots, m-1$, it holds that $|t_1 - t_2| < \delta$ implies that

$$\begin{aligned} &\left| (\mathcal{T}u)^{(i)}(t_1) - (\mathcal{T}u)^{(i)}(t_2) \right| \\ &\leq \lambda \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t_1, s) - \frac{\partial^i k}{\partial t^i}(t_2, s) \right| f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\leq \lambda \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t_1, s) - \frac{\partial^i k}{\partial t^i}(t_2, s) \right| \varphi_r(s) \, ds \\ &\leq \varepsilon \lambda \int_0^T \varphi_r(s) \, ds, \end{aligned}$$

and, since $\varphi_r \in L^1(I)$, it is clear that there exists a positive constant κ_1 such that

$$\left| (\mathcal{T}u)^{(i)}(t_1) - (\mathcal{T}u)^{(i)}(t_2) \right| < \kappa_1 \varepsilon$$

for all $u \in B$.

On the other hand, for the m -th derivative, for every $\varepsilon > 0$, take $\delta > 0$ given in

(H_1) and $|t_1 - t_2| < \delta$, $t_1 < t_2$, implies that

$$\begin{aligned}
 & \left| (\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2) \right| \\
 & \leq \lambda \int_0^T \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| f(s, u(s), \dots, u^{(m)}(s)) \, ds \\
 & \leq \lambda \int_0^T \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| \varphi_r(s) \, ds \\
 & = \lambda \int_0^{t_1} \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| \varphi_r(s) \, ds \\
 & \quad + \lambda \int_{t_1}^{t_2} \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| \varphi_r(s) \, ds \\
 & \quad + \lambda \int_{t_2}^T \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| \varphi_r(s) \, ds.
 \end{aligned}$$

From (H_1) , it is clear that first and third integrals in last term of previous expression can be arbitrarily small when $|t_1 - t_2| < \delta$. Moreover,

$$\left| \frac{\partial^m k}{\partial t^m}(t_1, \cdot) - \frac{\partial^m k}{\partial t^m}(t_2, \cdot) \right| \varphi_r(\cdot) \in L^1[t_1, t_2],$$

and so there exists some $\delta' > 0$ such that

$$\lambda \int_{t_1}^{t_2} \left| \frac{\partial^m k}{\partial t^m}(t_1, s) - \frac{\partial^m k}{\partial t^m}(t_2, s) \right| \varphi_r(s) \, ds < \varepsilon$$

when $|t_1 - t_2| < \delta'$.

Therefore it is clear that, for $|t_1 - t_2| < \min\{\delta, \delta'\}$, $t_1 < t_2$, there exists a positive constant κ_2 such that

$$\left| (\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2) \right| < \kappa_2 \varepsilon$$

for all $u \in B$.

Analogously, when $|t_1 - t_2| < \delta$, $t_1 > t_2$, there exists some positive constant κ_3 such that

$$\left| (\mathcal{T}u)^{(m)}(t_1) - (\mathcal{T}u)^{(m)}(t_2) \right| < \kappa_3 \varepsilon$$

for all $u \in B$.

We have proved the pointwise equicontinuity on I . Moreover, since I is compact, pointwise equicontinuity is equivalent to uniform equicontinuity, as it is stated in [129, Page 30 and Problem 31].

This way, we conclude that $\mathcal{T}(B)$ is equicontinuous in $\mathcal{C}^m(I)$.

As a consequence, by Ascoli-Arzelà's Theorem (Theorem 1.2.2), we can affirm that $\mathcal{T}(B)$ is relatively compact in $\mathcal{C}^m(I)$ and so \mathcal{T} is a compact operator. \square

6.3. Main Results

We introduce now the following notation

$$\Lambda^i := \int_0^T h_i(s) \, ds, \quad \Lambda_i := \int_{a_i}^{b_i} \xi_i \phi_i(s) \, ds$$

and define

$$\bar{\Lambda} := (m+1) \max\{\Lambda^i : i \in J\} \quad \text{and} \quad \underline{\Lambda} := \max\{\xi_i \Lambda_i : i \in J_1\}.$$

Moreover, we denote

$$f_0 := \liminf_{|x_0|, \dots, |x_m| \rightarrow 0} \min_{t \in I} \frac{f(t, x_0, \dots, x_m)}{|x_0| + \dots + |x_m|}$$

and

$$f^\infty := \limsup_{|x_0|, \dots, |x_m| \rightarrow \infty} \max_{t \in I} \frac{f(t, x_0, \dots, x_m)}{|x_0| + \dots + |x_m|}.$$

We will give now our existence result.

Theorem 6.3.1. *Assume that hypotheses (H_1) – (H_7) hold. If $\bar{\Lambda} f^\infty < \underline{\Lambda} f_0$, then for all*

$$\lambda \in \left(\frac{1}{\underline{\Lambda} f_0}, \frac{1}{\bar{\Lambda} f^\infty} \right)$$

operator \mathcal{T} has a fixed point in the cone K .

Proof. Let $\lambda \in \left(\frac{1}{\underline{\Lambda} f_0}, \frac{1}{\bar{\Lambda} f^\infty} \right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\underline{\Lambda} (f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\bar{\Lambda} (f^\infty + \varepsilon)}.$$

Taking into account the definition of f_0 , we know that there exists $\delta_1 > 0$ such that when $\|u\| \leq \delta_1$,

$$f(t, u(t), \dots, u^{(m)}(t)) > (f_0 - \varepsilon) \left(|u(t)| + \dots + |u^{(m)}(t)| \right), \quad \forall t \in I.$$

Let

$$\Omega_{\delta_1} = \{u \in K : \|u\| < \delta_1\}$$

and choose $u \in \partial \Omega_{\delta_1}$. We will prove that $\mathcal{T}u \not\leq u$.

Using (H_4) , we have that for $j \in J_1$ and $t \in [a_j, b_j] \cap [c_j, d_j]$,

$$\begin{aligned} (\mathcal{T}u)^{(j)}(t) &= \lambda \int_0^T \frac{\partial^j k}{\partial t^j}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\geq \lambda \int_{a_j}^{b_j} \frac{\partial^j k}{\partial t^j}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\geq \lambda \int_{a_j}^{b_j} \xi_j \phi_j(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &> \lambda \int_{a_j}^{b_j} \xi_j \phi_j(s) (f_0 - \varepsilon) (|u(s)| + \dots + |u^{(m)}(s)|) \, ds \\ &\geq \lambda (f_0 - \varepsilon) \xi_j \|u^{(j)}\|_{[c_j, d_j]} \int_{a_j}^{b_j} \xi_j \phi_j(s) \, ds \\ &= \lambda (f_0 - \varepsilon) \xi_j \|u^{(j)}\|_{[c_j, d_j]} \Lambda_j \geq \lambda (f_0 - \varepsilon) \xi_j \Lambda_j u^{(j)}(t). \end{aligned}$$

Now, for $j_0 \in J_1$ such that $\xi_{j_0} \Lambda_{j_0} = \underline{\Lambda}$, it holds that

$$(\mathcal{T}u)^{(j_0)}(t) > u^{(j_0)}(t) \text{ for all } t \in [a_{j_0}, b_{j_0}] \cap [c_{j_0}, d_{j_0}],$$

and so it is proved that $\mathcal{T}u \not\leq u$.

From Corollary 1.2.10, we deduce that

$$i_K(\mathcal{T}, \Omega_{\delta_1}) = 0.$$

On the other hand, due to the definition of f^∞ , we know that there exists $\tilde{C} > 0$ such that when $\min \{|u^{(i)}(t)| : i \in J\} \geq \tilde{C}$,

$$\begin{aligned} f(t, u(t), \dots, u^{(m)}(t)) &\leq (f^\infty + \varepsilon) (|u(t)| + \dots + |u^{(m)}(t)|) \\ &\leq (m+1) (f^\infty + \varepsilon) \|u\|, \end{aligned}$$

for all $t \in I$.

Let $C > \{\delta_1, \tilde{C}\}$ and define

$$\Omega_C = \bigcup_{i=0}^m \left\{ u \in K : \min_{t \in I} |u^{(i)}(t)| < C \right\}.$$

We note that Ω_C is an unbounded subset of the cone K . Because of this, the fixed point index of operator \mathcal{T} with respect to Ω_C , $i_K(\mathcal{T}, \Omega_C)$, is only defined in the case that the set of fixed points of operator \mathcal{T} in Ω_C , that is, $(I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$, is compact (see Section 1.2 for the details). We will see that $i_K(\mathcal{T}, \Omega_C)$ can be defined in this case.

First of all, since $(I - \mathcal{T})$ is a continuous operator, it is obvious that the set $(I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is closed.

Moreover, we can assume that $(I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is bounded. Indeed, on the contrary, we would have infinite fixed points of operator \mathcal{T} on Ω_C and it would be immediately deduced that \mathcal{T} has an infinite number of fixed points in the cone K . Therefore, we may assume that there exists a constant $M > 0$ such that $\|u\| < M$ for all $u \in (I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$.

Finally, it is left to see that $(I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is equicontinuous. This property follows from the fact that $(I - \mathcal{T})^{-1}(\{0\}) \cap \Omega_C$ is bounded. The proof is analogous to Step 3 in the proof of Lemma 6.2.2.

Now, we will calculate $i_K(\mathcal{T}, \Omega_C)$. In particular, we will prove that $\|\mathcal{T}u\| \leq \|u\|$ for all $u \in \partial\Omega_C$. Let $u \in \partial\Omega_C$, that is, $u \in K$ is such that

$$\min \left\{ \min_{t \in I} |u^{(i)}(t)| : i \in J \right\} = C.$$

Then, for $i \in J$,

$$\begin{aligned} |(\mathcal{T}u)^{(i)}(t)| &\leq \lambda \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\leq \lambda \int_0^T h_i(s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\leq (m+1) \lambda \int_0^T h_i(s) (f^\infty + \varepsilon) \|u\| \, ds \\ &= (m+1) \lambda (f^\infty + \varepsilon) \|u\| \Lambda^i \leq \lambda (f^\infty + \varepsilon) \|u\| \bar{\Lambda} \leq \|u\|. \end{aligned}$$

We deduce that

$$\|\mathcal{T}u\| \leq \|u\|$$

and, as a consequence of Corollary 1.2.9, we have that

$$i_K(\mathcal{T}, \Omega_C) = 1.$$

Therefore, we conclude from Lemma 1.2.7 that \mathcal{T} has a fixed point in $\bar{\Omega}_C \setminus \Omega_{\delta_1}$. \square

Consequently, we obtain the following corollary.

Corollary 6.3.2. *Assume that hypotheses (H_1) – (H_7) hold. Then,*

- (i) *If $f_0 = \infty$ and $f^\infty = 0$, then for all $\lambda \in (0, \infty)$, \mathcal{T} has a fixed point in the cone K .*
- (ii) *If $f_0 = \infty$ and $0 < f^\infty < \infty$, then for all $\lambda \in \left(0, \frac{1}{\Lambda f^\infty}\right)$, \mathcal{T} has a fixed point in the cone K .*
- (iii) *If $0 < f_0 < \infty$ and $f^\infty = 0$, then for all $\lambda \in \left(\frac{1}{\Lambda f_0}, \infty\right)$, \mathcal{T} has a fixed point in the cone K .*

6.4. Existence and Multiplicity of Solutions

In this section we will use the fixed point index theory to study the existence of multiple fixed points of operator \mathcal{T} . Similar arguments can be found in [28, 29, 79, 81, 149, 152].

We introduce now the following sets:

$$K_\rho = \{u \in K : \|u\| < \rho\},$$

$$V_\rho = \left\{ u \in K : \min_{t \in [a_i, b_i]} u^{(i)}(t) < \rho, i \in J_2, \|u^{(i)}\|_\infty < \rho, i \in J \setminus J_2 \right\},$$

where $J = \{0, \dots, m\}$ and

$$J_2 = \{i \in J : [c_i, d_i] = I\}.$$

To ensure that the sets K_ρ and V_ρ are not the same, we need to change condition (H_5) into

(\tilde{H}_5) There exists some index $i_0 \in \{0, \dots, m\}$ such that $[c_{i_0}, d_{i_0}] = I$ and, moreover, $\{0, 1, \dots, i_0\} \subset J_0$.

In this situation, it is clear that $J_2 \neq \emptyset$ and therefore

$$K_\rho \subsetneq V_\rho \subsetneq K_c^\rho,$$

where

$$c = \min\{\xi_i : i \in J_2\}. \quad (6.4.1)$$

Now we will give sufficient conditions under which the index of the previous sets is either 1 or 0.

Lemma 6.4.1. *Let*

$$\frac{1}{N} = \max \left\{ \sup_{t \in I} \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| ds : i \in J \right\}$$

and

$$f^\rho = \sup \left\{ \frac{f(t, x_0, \dots, x_m)}{\rho} : t \in I, x_i \in [0, \rho] \text{ if } [m_i, n_i] = [0, T], \right. \\ \left. x_i \in [-\rho, \rho] \text{ if } [m_i, n_i] \neq [0, T] \right\}.$$

If there exists $\rho > 0$ such that

$$\lambda \frac{f^\rho}{N} < 1, \quad (I_\rho^1)$$

then $i_K(\mathcal{T}, K_\rho) = 1$.

Proof. We will prove that $\mathcal{T}u \neq \mu u$ for all $u \in \partial K_\rho$ and for every $\mu \geq 1$.

Suppose, on the contrary, that there exist some $u \in \partial K_\rho$ and $\mu \geq 1$ such that

$$\mu u^{(i)}(t) = \lambda \int_0^T \frac{\partial^i k}{\partial t^i}(t, s) f(s, u(s), \dots, u^{(m)}(s)) ds.$$

Taking the supremum on I , we obtain that

$$\begin{aligned} \mu \|u^{(i)}\|_\infty &\leq \lambda \sup_{t \in I} \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| f(s, u(s), \dots, u^{(m)}(s)) ds \\ &\leq \lambda \rho f^\rho \sup_{t \in I} \int_0^T \left| \frac{\partial^i k}{\partial t^i}(t, s) \right| ds \leq \lambda \rho \frac{f^\rho}{N} < \rho. \end{aligned}$$

Consequently, we deduce that

$$\mu \rho = \mu \max \{ \|u^{(i)}\|_\infty : i \in J \} < \rho,$$

which contradicts the assumption that $\mu \geq 1$. Therefore, by Lemma 1.2.7, we conclude that $i_K(\mathcal{T}, K_\rho) = 1$. \square

Lemma 6.4.2. *For $i \in J_1$, let*

$$\frac{1}{M_i} = \inf_{t \in [a_i, b_i]} \int_{a_i}^{b_i} \frac{\partial^i k}{\partial t^i}(t, s) ds,$$

and

$$f_\rho^i = \inf \left\{ \frac{f(t, x_0, \dots, x_m)}{\rho} : t \in [a_i, b_i], x_j \in \left[0, \frac{\rho}{\xi_j}\right], j \in J_2, \right. \\ \left. x_k \in [0, \rho], k \in J \setminus J_2 \right\}.$$

If there exists $\rho > 0$ and $i_0 \in J_1$ such that

$$\lambda \frac{f_{\rho}^{i_0}}{M_{i_0}} > 1, \quad (I_{\rho}^0)$$

then $i_K(\mathcal{T}, V_{\rho}) = 0$.

Proof. We will prove that there exists $e \in K \setminus \{0\}$ such that $u \neq \mathcal{T}u + \alpha e$ for all $u \in \partial V_{\rho}$ and all $\alpha > 0$.

Let us take $e(t) = 1$ and suppose that there exists some $u \in \partial V_{\rho}$ and $\alpha > 0$ such that $u = \mathcal{T}u + \alpha$. Then, for $t \in [a_{i_0}, b_{i_0}]$,

$$\begin{aligned} u^{(i_0)}(t) &\geq \lambda \int_0^T \frac{\partial^{i_0} k}{\partial t^{i_0}}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\geq \lambda \int_{a_{i_0}}^{b_{i_0}} \frac{\partial^{i_0} k}{\partial t^{i_0}}(t, s) f(s, u(s), \dots, u^{(m)}(s)) \, ds \\ &\geq \lambda \rho f_{\rho}^{i_0} \int_{a_{i_0}}^{b_{i_0}} \frac{\partial^{i_0} k}{\partial t^{i_0}}(t, s) \, ds > \rho. \end{aligned}$$

Consequently, $u^{(i_0)}(t) > \rho$ for $t \in [a_{i_0}, b_{i_0}]$, which is a contradiction and, therefore, by Lemma 1.2.7, $i_K(\mathcal{T}, V_{\rho}) = 0$. \square

Combining the previous lemmas, it is possible to obtain some conditions under which operator \mathcal{T} has multiple fixed points.

Theorem 6.4.3. Assume that conditions (H_1) – (H_4) , (\tilde{H}_5) and (H_6) – (H_7) hold, and let c be defined in (6.4.1). The integral equation (6.1.1) has at least one non trivial solution in K if one of the following conditions holds:

(C1) There exist $\rho_1, \rho_2 \in (0, \infty)$, $\frac{\rho_1}{c} < \rho_2$, such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ are satisfied.

(C2) There exist $\rho_1, \rho_2 \in (0, \infty)$, $\rho_1 < \rho_2$, such that $(I_{\rho_1}^1)$ and $(I_{\rho_2}^0)$ are satisfied.

The integral equation (6.1.1) has at least two non trivial solutions in K if one of the following conditions holds:

(C3) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, $\frac{\rho_1}{c} < \rho_2 < \rho_3$, such that $(I_{\rho_1}^0)$, $(I_{\rho_2}^1)$ and $(I_{\rho_3}^0)$ are satisfied.

(C4) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2$ and $\frac{\rho_2}{c} < \rho_3$, such that $(I_{\rho_1}^1)$, $(I_{\rho_2}^0)$ and $(I_{\rho_3}^1)$ are satisfied.

The integral equation (6.1.1) has at least three non trivial solutions in K if one of the following conditions holds:

(C5) There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$, with $\frac{\rho_1}{c} < \rho_2 < \rho_3$ and $\frac{\rho_3}{c} < \rho_4$, such that $(I_{\rho_1}^0), (I_{\rho_2}^1), (I_{\rho_3}^0)$ and $(I_{\rho_4}^1)$ are satisfied.

(C6) There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$, with $\rho_1 < \rho_2$ and $\frac{\rho_2}{c} < \rho_3 < \rho_4$, such that $(I_{\rho_1}^1), (I_{\rho_2}^0), (I_{\rho_3}^1)$ and $(I_{\rho_4}^0)$ are satisfied.

The proof of the previous result is an immediate consequence of the properties of the fixed point index given in Lemma 1.2.7. Moreover, it must be point out that, despite of the fact that the previous theorem studies the existence of one, two or three fixed points, similar results can be formulated to ensure the existence of four or more fixed points.

6.5. Examples

In this section we will show two examples in which the theory previously developed will be applied. In particular, these examples will show that the existence results given in Theorems 6.3.1 and 6.4.3 are not comparable.

Example 6.5.1. Consider the following boundary value problem:

$$\begin{cases} u^{(3)}(t) = \lambda \frac{e^t (|u(t)| + |u'(t)| + |u''(t)|)}{1 + (u(t))^2}, & t \in [0, 1], \\ u(0) = -u(1), \quad u'(0) = \frac{1}{2} u'(1), \quad u''(0) = 0. \end{cases} \quad (6.5.1)$$

The Green's function related to the homogeneous problem

$$\begin{cases} u^{(3)}(t) = 0, & t \in [0, 1], \\ u(0) = -u(1), \quad u'(0) = \frac{1}{2} u'(1), \quad u''(0) = 0, \end{cases}$$

which has been calculated using [24], is the following one

$$G(t, s) = \begin{cases} \frac{1}{4} (1 - s) (-3 + s + 4t), & t \leq s, \\ \frac{1}{4} (-3 + s(s + 4) + 2t(t + 2) - 8st), & s < t. \end{cases}$$

Therefore, solutions of boundary value problem (6.5.1) correspond with the fixed points of the following operator:

$$\mathcal{T}u(t) = \lambda \int_0^1 G(t, s) f(s, u(s), u'(s), u''(s)) \, ds, \quad t \in [0, 1],$$

which is a particular case of the operator defined in (6.1.1) for $T = 1$, $m = 2$, $k \equiv G$ and

$$f(t, x, y, z) = \frac{e^t (|x| + |y| + |z|)}{1 + x^2}.$$

We will check now that the kernel G satisfies conditions (H_1) – (H_5) . To do that, we need to calculate the explicit expression of the first and second order partial derivatives of the Green's function, that is,

$$\frac{\partial G}{\partial t}(t, s) = \begin{cases} 1 - s, & t \leq s, \\ 1 - 2s + t, & s < t, \end{cases}$$

and

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \begin{cases} 0, & t < s, \\ 1, & s < t. \end{cases}$$

Using these expressions, we are able to check that the required conditions hold:

(H_1) Let $\tau \in I$ be fixed. Both G and $\frac{\partial G}{\partial t}$ are uniformly continuous, so the hypothesis is immediate for $i = 0, 1$. Moreover, for the second derivative $\frac{\partial^2 G}{\partial t^2}$ (that is, for the case $i = m = 2$), we can take $Z_\tau = \{\tau\}$ and we have that

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) - \frac{\partial^2 G}{\partial t^2}(\tau, s) \right| = |1 - 1| = 0, \quad \forall s < \min\{t, \tau\}$$

and

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) - \frac{\partial^2 G}{\partial t^2}(\tau, s) \right| = |0 - 0| = 0, \quad \forall s > \max\{t, \tau\},$$

so the hypothesis holds.

(H_2) It can be seen that

$$G(t, s) \geq 0, \quad \text{for all } t \in [t_0, 1], \quad s \in [0, 1],$$

with $t_0 \approx 0.6133$. Therefore, in this case $[m_0, n_0] = [t_0, 1]$.

Moreover, both $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial t^2}$ are nonnegative on the square $[0, 1] \times [0, 1]$, which means that $[m_1, n_1] = [m_2, n_2] = [0, 1]$.

(H_3) It can be checked that

$$|G(t, s)| \leq \frac{1}{4} (3 - 4s + s^2), \quad \text{for all } t \in [0, 1], \quad s \in [0, 1],$$

and the equality holds for $t = 0$ and $t = 1$, so the choice $h_0(s) = \frac{1}{4}(3 - 4s + s^2)$ is optimal. This inequality can be easily proved by taking into account that, since $\frac{\partial G}{\partial t}$ is nonnegative, then $G(\cdot, s)$ is nondecreasing for every $s \in [0, 1]$ and, therefore,

$$|G(t, s)| \leq \max\{|G(0, s)|, |G(1, s)|\} = \frac{1}{4}(3 - 4s + s^2).$$

For the first derivative, it occurs that

$$\left| \frac{\partial G}{\partial t}(t, s) \right| \leq 2(1 - s), \text{ for all } t \in [0, 1], s \in [0, 1],$$

and the equality holds for $t = 1$, so $h_1(s) = 2(1 - s)$ is also optimal.

Finally,

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| \leq 1, \text{ for } t \in [0, 1] \text{ and a. e. } s \in [0, 1],$$

and $h_2(s) = 1$ is trivially optimal.

(H₄) If we take $\phi_0(s) = h_0(s) = \frac{1}{4}(3 - 4s + s^2)$, $[c_0, d_0] = [0, 1]$, and $[a_0, b_0] = [t_1, 1]$ with $t_1 > t_0$ (t_0 given in (H₂)), it holds that there exists a constant $\xi_0(t_1) \in (0, 1)$ such that

$$G(t, s) \geq \xi_0(t_1) \phi_0(s), \text{ for all } t \in [t_1, 1], s \in [0, 1].$$

We note that the bigger t_1 is, the bigger the constant $\xi_0(t_1)$ is. For instance, if we take $t_1 = 0.62$, we can choose $\xi_0 = \frac{1}{75}$.

With regard to the first derivative of G , it satisfies that

$$\frac{\partial G}{\partial t}(t, s) \leq 2(1 - s), \text{ for all } t \in [0, 1], s \in [0, 1],$$

and

$$\frac{\partial G}{\partial t}(t, s) \geq 1 - s, \text{ for all } t \in [0, 1], s \in [0, 1],$$

that is, we could take $\phi_1(s) = h_1(s) = 2(1 - s)$, $[c_1, d_1] = [0, 1]$, $\xi_1 = \frac{1}{2}$ and $[a_1, b_1] = [0, 1]$.

Finally, for the second derivative of G , it does not exist a suitable function ϕ_2 and a constant ξ_2 for which the inequalities in (H₄) hold.

As a consequence, we deduce that $J_1 = \{0, 1\}$.

Moreover, it is obvious that $\int_{a_i}^{b_i} \phi_i(s) \, ds > 0$ for $i = 0, 1$.

(H₅) It is immediately deduced from the proofs of the previous conditions.

Moreover, the nonlinearity f satisfies condition (H₆), being $\varphi_r(t) = 3r e^t$. Finally, it is clear that condition (H₇) also holds.

We will work in the cone

$$K = \left\{ u \in \mathcal{C}^2([0, 1], \mathbb{R}) : u(t) \geq 0, t \in [t_0, 1], \quad u'(t), u''(t) \geq 0, t \in [0, 1]; \right. \\ \left. \min_{t \in [t_1, 1]} u(t) \geq \xi_0(t_1) \|u\|_{[0, 1]}, \quad \min_{t \in [0, 1]} u'(t) \geq \frac{1}{2} \|u'\|_{[0, 1]} \right\}.$$

With the notation introduced in Section 6.3, we obtain the following values for the constants involved in Theorem 6.3.1:

$$\Lambda^0 = \frac{1}{3}, \quad \Lambda^1 = 1, \quad \Lambda^2 = 1,$$

and therefore

$$\bar{\Lambda} = 3 \max \{ \Lambda^0, \Lambda^1, \Lambda^2 \} = 3,$$

$$\Lambda_0 = \xi_0(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right), \quad \Lambda_1 = \frac{1}{2},$$

and so

$$\underline{\Lambda} = \max \left\{ \xi_0^2(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right), \frac{1}{4} \right\}.$$

We note that, since $\xi_0(t_1) \in (0, 1)$,

$$\xi_0^2(t_1) \left(\frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3 \right) < \frac{1}{3} - \frac{3}{4} t_1 + \frac{1}{2} t_1^2 - \frac{1}{12} t_1^3$$

and it is easy to see that the right hand side of previous inequality decreases with t_1 and, in particular, it is always smaller than $\frac{1}{4}$. Thus,

$$\underline{\Lambda} = \frac{1}{4}$$

independently of the value of t_1 .

On the other hand, we obtain the following values for the limits over the nonlinearity f :

$$f_0 = \liminf_{|x|, |y|, |z| \rightarrow 0} \min_{t \in [0, 1]} \frac{e^t (|x| + |y| + |z|)}{(1 + x^2) (|x| + |y| + |z|)} = \lim_{|x|, |y|, |z| \rightarrow 0} \frac{1}{(1 + x^2)} = 1,$$

$$f^\infty = \limsup_{|x|, |y|, |z| \rightarrow \infty} \max_{t \in [0, 1]} \frac{e^t (|x| + |y| + |z|)}{(1 + x^2) (|x| + |y| + |z|)} = \lim_{|x|, |y|, |z| \rightarrow \infty} \frac{e}{(1 + x^2)} = 0.$$

Therefore, from Corollary 6.3.2, we deduce that for all $\lambda > 4$, \mathcal{T} has at least a fixed point in the cone K . This fixed point is a nontrivial solution of problem (6.5.1).

On the other hand, we will prove that it is not possible to apply Theorem 6.4.3 to this example. With the notation introduced in Lemma 6.4.2, we have that

$$f_{\rho}^0 = \inf \left\{ \frac{e^t(|x| + |y| + |z|)}{\rho(x^2 + 1)} : t \in [t_1, 1], x \in \left[0, \frac{\rho}{\xi_0(t_1)}\right], \right. \\ \left. y \in [0, 2\rho], z \in [0, \rho] \right\} = 0$$

and

$$f_{\rho}^1 = \inf \left\{ \frac{e^t(|x| + |y| + |z|)}{\rho(x^2 + 1)} : t \in [0, 1], x \in \left[0, \frac{\rho}{\xi_0(t_1)}\right], \right. \\ \left. y \in [0, 2\rho], z \in [0, \rho] \right\} = 0,$$

and therefore it does not exist any ρ such that condition (I_{ρ}^0) holds. Thus Theorem 6.4.3 is not applicable to this example.

Example 6.5.2. Consider now the following fourth order Dirichlet problem:

$$\begin{cases} u^{(4)}(t) = \lambda t \left(e^{u(t)} + (u'(t))^2 + (u''(t))^2 + (u'''(t))^2 \right), & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (6.5.2)$$

Notice that fourth order differential equations with this type of boundary conditions have been applied for the study of the bending of simply supported elastic beams (see [115, 141]) or suspension bridges (see [50, 97]).

The Green's function related to the homogeneous problem

$$\begin{cases} u^{(4)}(t) = 0, & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

has the following expression:

$$G(t, s) = \frac{1}{6} \begin{cases} t(1-s)(2s-s^2-t^2), & t \leq s, \\ s(1-t)(2t-t^2-s^2), & s < t, \end{cases}$$

which implies that the solutions of problem (6.5.2) coincide with the fixed points of

$$\mathcal{T}u(t) = \lambda \int_0^1 G(t, s) f(s, u(s), u'(s), u''(s), u'''(s)) \, ds, \quad t \in [0, 1].$$

Previous operator is a particular case of (6.1.1) for $T = 1$, $m = 3$, $k \equiv G$ and $f(t, x, y, z, w) = t (e^x + y^2 + z^2 + w^2)$.

Next, we will give the explicit expressions of the first, second and third derivatives of the Green's function:

$$\frac{\partial G}{\partial t}(t, s) = \frac{1}{6} \begin{cases} -(1-s)(-2s + s^2 + 3t^2), & t \leq s, \\ s(2 + s^2 + 3t^2 - 6t), & s < t, \end{cases}$$

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \begin{cases} -t(1-s), & t \leq s, \\ -s(1-t), & s < t, \end{cases}$$

$$\frac{\partial^3 G}{\partial t^3}(t, s) = \begin{cases} -(1-s), & t < s, \\ s, & s < t, \end{cases}$$

and now we will see that they satisfy the required hypotheses:

(H₁) As in previous example, it is easy to verify that this condition holds.

(H₂) The Green's function G is nonnegative on $[0, 1] \times [0, 1]$ (in fact it is positive on $(0, 1) \times (0, 1)$). Therefore $[m_0, n_0] = [0, 1]$.

For first derivative it holds that

$$\frac{\partial G}{\partial t}(t, s) \geq 0 \quad \text{for all } t \in [0, t_2], s \in [0, 1],$$

with $t_2 = 1 - \frac{\sqrt{3}}{3} \approx 0.42265$. Thus $[m_1, n_1] = [0, t_2]$.

With respect to the second derivative, it is immediate to see that it is nonpositive on its square of definition. However it is zero on the boundary of the square, so we could take $[m_2, n_2] = \{0\}$ (note that it would also be possible to choose $[m_2, n_2] = \{1\}$).

Finally, the third derivative is nonnegative on the triangle

$$\{(t, s) \in [0, 1] \times [0, 1] : s < t\},$$

that is, $[m_3, n_3] = \{1\}$.

(H₃) We have that

$$|G(t, s)| = G(t, s) \leq h_0(s) \quad \text{for all } t \in [0, 1], s \in [0, 1],$$

where

$$h_0(s) = \frac{1}{9\sqrt{3}} \begin{cases} s(1-s^2)^{\frac{3}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ (1-s)(2s-s^2)^{\frac{3}{2}}, & \frac{1}{2} < s \leq 1. \end{cases}$$

The previous inequality has been proved in [150] but we include the proof for the sake of completeness: we note that G is zero at the boundary of its square of definition so, since it is nonnegative, it is clear that the restricted function $G(\cdot, s)$ will attain its maximum on a point in which $\frac{\partial G}{\partial t}$ is null. It is immediate to check that:

- For $s \leq \frac{1}{2}$, it holds that $\frac{\partial G}{\partial t}(t, s) = 0$ if and only if $t = \frac{1}{3}(3 - \sqrt{3(1-s^2)})$ and $t \leq \frac{1}{2}$.
- For $s \geq \frac{1}{2}$, it holds that $\frac{\partial G}{\partial t}(t, s) = 0$ if and only if $t = \frac{\sqrt{2s-s^2}}{\sqrt{3}}$ and $t \geq \frac{1}{2}$.

Therefore, due to the regularity properties of the Green's function, since $\frac{\partial^2 G}{\partial t^2}$ is nonpositive at every point of its square of definition, we deduce that:

- For $s \leq \frac{1}{2}$, $G(\cdot, s)$ has its maximum at $\left(\frac{1}{3}(3 - \sqrt{3(1-s^2)}), s\right)$. In particular,

$$G\left(\frac{1}{3}(3 - \sqrt{3(1-s^2)}), s\right) = \frac{1}{9\sqrt{3}} s(1-s^2)^{\frac{3}{2}}.$$

- For $s \geq \frac{1}{2}$, $G(\cdot, s)$ has its maximum at $\left(\frac{\sqrt{2s-s^2}}{\sqrt{3}}, s\right)$. In particular

$$G\left(\frac{\sqrt{2s-s^2}}{\sqrt{3}}, s\right) = \frac{1}{9\sqrt{3}} (1-s)(2s-s^2)^{\frac{3}{2}}.$$

Thus, it is clear that

$$G(t, s) \leq h_0(s), \quad \text{for all } t \in [0, 1], s \in [0, 1].$$

Previous inequality is optimal in the sense that, for each $s \in [0, 1]$, there exists at least one value of $t \in [0, 1]$ for which the equality is satisfied.

Analogously, it holds that

$$\left| \frac{\partial G}{\partial t}(t, s) \right| \leq h_1(s) \quad \text{for all } t \in [0, 1], s \in [0, 1],$$

for

$$h_1(s) = \frac{1}{6} s(1-s) \begin{cases} 2-s, & 0 \leq s \leq \frac{1}{2}, \\ 1+s, & \frac{1}{2} < s \leq 1, \end{cases}$$

and the equality holds for $0 \leq s \leq \frac{1}{2}$ at $t = 0$ and for $\frac{1}{2} < s \leq 1$ at $t = 1$, so this choice of h_1 is optimal. This inequality is easily proved by just taking into account the fact that $\frac{\partial^2 G}{\partial t^2}$ is nonpositive and so $\frac{\partial G}{\partial t}(\cdot, s)$ is decreasing for every $s \in [0, 1]$. Therefore,

$$\begin{aligned} \left| \frac{\partial G}{\partial t}(t, s) \right| &\leq \max \left\{ \left| \frac{\partial G}{\partial t}(0, s) \right|, \left| \frac{\partial G}{\partial t}(1, s) \right| \right\} \\ &= \frac{1}{6} s(1-s) \max \{2-s, 1+s\} = h_1(s). \end{aligned}$$

For the second derivative, we have that

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| \leq s(1-s) \equiv h_2(s) \quad \text{for all } t \in [0, 1], s \in [0, 1],$$

and the inequality is optimal in the same way it was for the Green's function G .

With regard to the third derivative, it satisfies that

$$\left| \frac{\partial^3 G}{\partial t^3}(t, s) \right| \leq \max\{s, 1-s\} \equiv h_3(s) \quad \text{for } t \in [0, 1] \text{ and a.e. } s \in [0, 1],$$

and the inequality is also optimal.

(H₄) If we choose $\phi_0(s) = h_0(s)$, given in (H₃), and $[c_0, d_0] = [0, 1]$ then, for any closed interval $[a_0, b_0] \subset (0, 1)$, it is possible to find a constant $\xi_0(a_0, b_0) \in (0, 1)$ such that

$$G(t, s) \geq \xi_0(a_0, b_0) \phi_0(s), \quad \text{for all } t \in [a_0, b_0], s \in [0, 1].$$

This has been proved in [150] with an explicit function. Of course, it is satisfied that the bigger the interval $[a_0, b_0]$ is, the smaller $\xi_0(a_0, b_0)$ needs to be.

Analogously, we can take $\phi_1(s) = h_1(s)$ and $[c_1, d_1] = [0, 1]$ and it holds that for any interval $[0, b_1]$, with $b_1 < 1 - \frac{\sqrt{3}}{3}$, there exists $\xi_1(b_1) \in (0, 1)$ such that

$$\frac{\partial G}{\partial t}(t, s) \geq \xi_1(b_1) \phi_1(s), \quad \text{for all } t \in [0, b_1], s \in [0, 1].$$

Finally, with respect to the second derivative of the Green's function G , it does not exist any pair of a function ϕ_2 and a constant ξ_2 such that the inequalities in (H₄) hold. The same occurs with the third derivative of G . Therefore, $J_1 = J_2 = \{0, 1\}$.

(\tilde{H}_5) It is a direct consequence of (H_4).

Clearly, f satisfies (H_6) and (H_7), being $\varphi_r(t) = t(e^r + 3r^2)$.

As a consequence of the properties of the Green's function that we have just seen, we will work in the cone

$$K = \left\{ u \in C^3([0, 1], \mathbb{R}) : \begin{aligned} &u(t) \geq 0, t \in [0, 1], \quad u'(t) \geq 0, t \in [0, t_2], \\ &u''(t) \geq 0, t \in \{0, 1\}, \quad u'''(1) \geq 0, \\ &\min_{t \in [a_0, b_0]} u(t) \geq \xi_0(a_0, b_0) \|u\|_{[0, 1]}, \\ &\min_{t \in [0, b_1]} u'(t) \geq \xi_1(b_1) \|u'\|_{[0, 1]} \end{aligned} \right\}.$$

Moreover, we will make all the calculations with the values $[a_0, b_0] = [0.1, 0.9]$, $\xi_0 = \frac{1}{4}$, $[0, b_1] = [0, \frac{1}{3}]$ and $\xi_1 = \frac{1}{6}$.

In this case, with the notation introduced in Lemma 6.4.1, we have that

$$\frac{1}{N} = \max \left\{ \frac{5}{384}, \frac{1}{24}, \frac{1}{8}, \frac{1}{2} \right\} = \frac{1}{2}$$

and

$$\begin{aligned} f^{\rho_2} &= \sup \left\{ \frac{t(e^x + y^2 + z^2 + w^2)}{\rho_2} : t \in [0, 1], x, y, z, w \in [-\rho_2, \rho_2] \right\} \\ &= \frac{e^{\rho_2} + 3\rho_2^2}{\rho_2}, \end{aligned}$$

and so ($I_{\rho_2}^1$) holds for any

$$\lambda < \frac{2\rho_2}{e^{\rho_2} + 3\rho_2^2}.$$

Analogously, with the notation used in Lemma 6.4.2,

$$\frac{1}{M_0} = \frac{29}{7500}, \quad \frac{1}{M_1} = \frac{7}{1944},$$

$$f_{\rho_1}^0 = \inf \left\{ \frac{t(e^x + y^2 + z^2 + w^2)}{\rho_1} : \begin{aligned} &t \in [0.1, 0.9], x \in [0, 4\rho_1], \\ &y \in [0, 6\rho_1], z, w \in [0, \rho_1] \end{aligned} \right\} = \frac{0.1}{\rho_1},$$

and

$$f_{\rho}^1 = \inf \left\{ \frac{t(e^x + y^2 + z^2 + w^2)}{\rho_1} : t \in \left[0, \frac{1}{3}\right], x \in [0, 4\rho_1], \right. \\ \left. y \in [0, 6\rho_1], z, w \in [0, \rho_1] \right\} = 0,$$

and thus $(I_{\rho_1}^0)$ holds for

$$\lambda > \frac{75000 \rho_1}{29}.$$

Therefore, as a consequence of (C1) in Theorem 6.4.3, for any pair of values $\rho_1, \rho_2 > 0$ such that $\rho_1 < c \rho_2 = \frac{\rho_2}{6}$ and

$$\frac{75000 \rho_1}{29} < \frac{2 \rho_2}{e^{\rho_2} + 3 \rho_2^2},$$

problem (6.5.2) has at least a nontrivial solution for all

$$\lambda \in \left(\frac{75000 \rho_1}{29}, \frac{2 \rho_2}{e^{\rho_2} + 3 \rho_2^2} \right).$$

In particular, there exists at least a nontrivial solution of (6.5.2) for all

$$\lambda \in (0, 0.4171).$$

On the other hand, we obtain that:

$$f_0 = \liminf_{|x|, |y|, |z|, |w| \rightarrow 0} \min_{t \in [0, 1]} \frac{t(e^x + y^2 + z^2 + w^2)}{|x| + |y| + |z| + |w|} = 0,$$

and thus neither Theorem 6.3.1 nor Corollary 6.3.2 can be applied to this example.

6.6. Application to some Even Order Problems

In this section we will show an application of our results to solve some general $2n$ -th order Dirichlet problems. We note that these boundary conditions are also called Lidstone conditions in some references.

In particular, we will contribute to fill some gaps on the study of general $2n$ -th order Dirichlet boundary value problems for $n \geq 2$. We note that the case $n = 2$ has already been considered in Example 6.5.2.

Usually, in this kind of problems, the nonlinearities may depend only on the even derivatives (see, for example, [43, 86, 146, 159]). We will generalize now these studies

by considering the following problem, with a full nonlinearity (that is, depending on all the derivatives up to order $2n - 1$),

$$\begin{cases} u^{(2n)}(t) = f(t, u(t), \dots, u^{(2n-1)}(t)), & t \in [0, 1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, \dots, n-1. \end{cases} \quad (6.6.1)$$

Let $G(t, s)$ be the Green's function related to the homogeneous problem

$$\begin{cases} u^{(2n)}(t) = 0, & t \in [0, 1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, \dots, n-1. \end{cases}$$

It can be checked that, for $n \geq 2$, $g(t, s) = \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s)$ is the Green's function related to the problem

$$\begin{cases} u^{(4)}(t) = 0, & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

whose explicit expression has been calculated in Example 6.5.2. As a consequence of the calculations made in that example we know that the following facts hold for $n \geq 2$:

- $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) = g(t, s) \geq 0$ on $[0, 1] \times [0, 1]$ and $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) = 0$ on the boundary of the square.
- $\frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t, s) = \frac{\partial g}{\partial t}(t, s) \geq 0$ on $[0, t_2] \times [0, 1]$, with $t_2 = 1 - \frac{\sqrt{3}}{3}$.
- $\frac{\partial^{2n-2} G}{\partial t^{2n-2}}(t, s) = \frac{\partial^2 g}{\partial t^2}(t, s) \leq 0$ on $[0, 1] \times [0, 1]$, and $\frac{\partial^{2n-2} G}{\partial t^{2n-2}}(t, s) = 0$ on the boundary of the square.
- $\frac{\partial^{2n-1} G}{\partial t^{2n-1}}(t, s) = \frac{\partial^3 g}{\partial t^3}(1, s) \geq 0$ for $s \in [0, 1]$.

With this information, we can deduce some results about the constant sign both of the derivatives of smaller order of G and of the Green's function itself.

1. Since $\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) \geq 0$, for $n \geq 3$, it holds that for each fixed $s \in [0, 1]$, $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot, s)$ is nondecreasing.

Assume that it is nonnegative. Then $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(\cdot, s)$ would also be nondecreasing and, since the boundary value conditions imply that

$$\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(0, s) = \frac{\partial^{2n-6} G}{\partial t^{2n-6}}(1, s) = 0,$$

we would conclude that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(t, s) = 0$ for all $(t, s) \in [0, 1] \times [0, 1]$, which is not possible.

The same argument holds if we assume that $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot, s)$ is nonpositive.

Therefore, necessarily $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot, s)$ is sign-changing and, since it is nondecreasing, we know for sure that

$$\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(0, s) < 0 \text{ and } \frac{\partial^{2n-5} G}{\partial t^{2n-5}}(1, s) > 0 \text{ for all } s \in [0, 1] \text{ and } n \geq 3.$$

2. Now, since $\frac{\partial^{2n-5} G}{\partial t^{2n-5}}(\cdot, s)$ is sign-changing and nondecreasing, $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(\cdot, s)$ will be first decreasing and then increasing. This together with the boundary value conditions

$$\frac{\partial^{2n-6} G}{\partial t^{2n-6}}(0, s) = \frac{\partial^{2n-6} G}{\partial t^{2n-6}}(1, s) = 0$$

implies that $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}$ is nonpositive for $n \geq 3$.

3. Since $\frac{\partial^{2n-6} G}{\partial t^{2n-6}}$ is nonpositive, we can follow an analogous argument to the one made in 1. to deduce that $\frac{\partial^{2n-7} G}{\partial t^{2n-7}}$ is sign-changing and nonincreasing. In particular this implies that

$$\frac{\partial^{2n-7} G}{\partial t^{2n-7}}(0, s) > 0 \text{ and } \frac{\partial^{2n-7} G}{\partial t^{2n-7}}(1, s) < 0 \text{ for all } s \in [0, 1] \text{ and } n \geq 4.$$

4. Finally, arguing analogously to 2., we can deduce that $\frac{\partial^{2n-8} G}{\partial t^{2n-8}}$ is nonnegative on $[0, 1] \times [0, 1]$, for $n \geq 4$.

Note that all the previous arguments could be repeated iteratively. This way, the following sign-criteria for the derivatives of G can be deduced for $n \geq \frac{k}{2}$:

- If $k \equiv 0 \pmod{4}$, then

$$\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(t, s) \geq 0 \text{ on } [0, 1] \times [0, 1].$$

- If $k \equiv 1 \pmod{4}$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(\cdot, s)$ is sign-changing and nondecreasing for every $s \in [0, 1]$. In particular,

$$\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(0, s) < 0 \text{ and } \frac{\partial^{2n-k} G}{\partial t^{2n-k}}(1, s) > 0 \text{ for every } s \in [0, 1].$$

- If $k \equiv 2 \pmod{4}$, then

$$\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(t, s) \leq 0 \text{ on } [0, 1] \times [0, 1].$$

- If $k \equiv 3 \pmod{4}$, then $\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(\cdot, s)$ is sign-changing and nonincreasing for every $s \in [0, 1]$. In particular,

$$\frac{\partial^{2n-k} G}{\partial t^{2n-k}}(0, s) > 0 \text{ and } \frac{\partial^{2n-k} G}{\partial t^{2n-k}}(1, s) < 0 \text{ for every } s \in [0, 1].$$

In particular, if n is even, we could deduce that $G(t, s) \geq 0$ on $[0, 1] \times [0, 1]$ and, if n is odd, $G(t, s) \leq 0$ on $[0, 1] \times [0, 1]$.

Therefore, the Green's function and its derivatives satisfy the required hypotheses:

(H_1) As in Example 6.5.2, this condition holds as a direct consequence of the general properties of the Green's function.

(H_2) As we have just proved, we could take

$$\begin{aligned} [m_{2n-i}, n_{2n-i}] &= [0, 1] \text{ for } i \equiv 0 \pmod{4}, \\ [m_{2n-i}, n_{2n-i}] &= \{1\} \text{ for } i \equiv 1 \pmod{4}, \\ [m_{2n-i}, n_{2n-i}] &= \{0\} \text{ for } i \equiv 2 \pmod{4} \end{aligned}$$

and

$$[m_{2n-i}, n_{2n-i}] = \{0\} \text{ for } i \equiv 3 \pmod{4}.$$

(H_3) It is enough to take $h_i(s) = \max \left\{ \left| \frac{\partial^i G}{\partial t^i}(t, s) \right| : t \in [0, 1] \right\}$, for $i \in J$.

(H_4) For $n \geq 2$, we could take $J_1 = \{2n - 4, 2n - 3\}$. As a consequence of Example 6.5.2, we know that

$$\left| \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) \right| = \frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) = g(t, s) \leq \phi_{2n-4}(s),$$

with

$$\phi_{2n-4}(s) = \frac{1}{9\sqrt{3}} \begin{cases} s(1-s^2)^{\frac{3}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ (1-s)(2s-s^2)^{\frac{3}{2}}, & \frac{1}{2} < s \leq 1. \end{cases}$$

Moreover, it holds that for any closed interval $[a_{2n-4}, b_{2n-4}] \subset [0, 1]$, there exists a constant $\xi_{2n-4}(a_{2n-4}, b_{2n-4}) \in (0, 1)$ such that

$$\frac{\partial^{2n-4} G}{\partial t^{2n-4}}(t, s) \geq \xi_{2n-4}(a_{2n-4}, b_{2n-4}) \phi_{2n-4}(s),$$

for all $t \in [a_{2n-4}, b_{2n-4}]$, $s \in [0, 1]$. Obviously, the bigger the interval $[a_{2n-4}, b_{2n-4}]$ is, the bigger the constant $\xi_{2n-4}(a_{2n-4}, b_{2n-4})$ needs to be.

Analogously, from Example 6.5.2 we know that

$$\left| \frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t, s) \right| = \left| \frac{\partial g}{\partial t}(t, s) \right| \leq \phi_{2n-3}(s) = \frac{1}{6} s(1-s) \begin{cases} 2-s, & 0 \leq s \leq \frac{1}{2}, \\ 1+s, & \frac{1}{2} < s \leq 1, \end{cases}$$

for all $t \in [0, 1]$, $s \in [0, 1]$, and for any interval $[0, b_{2n-3}]$, with $b_{2n-3} < 1 - \frac{\sqrt{3}}{3}$ there exists $\xi_{2n-3}(b_{2n-3}) \in (0, 1)$ such that

$$\frac{\partial^{2n-3} G}{\partial t^{2n-3}}(t, s) \geq \xi_{2n-3}(b_{2n-3}) \phi_{2n-3}(s), \text{ for all } t \in [0, b_{2n-3}], s \in [0, 1].$$

Again, the bigger b_{2n-3} is, the bigger $\xi_{2n-3}(b_{2n-3})$ needs to be.

(H_5) As we have already seen, it holds that $[m_{2n-4}, n_{2n-4}] = [0, 1]$.

(\tilde{H}_5) As we have seen, it holds that $[c_{2n-3}, d_{2n-3}] = [0, 1]$ and, moreover, $J_0 = J$.

Then, for $n \in \mathbb{N}$ such that $n \geq \max \left\{ 2, \frac{i}{2} \right\}$, we could work in the cone

$$K = \left\{ u \in \mathcal{C}^{2n-1}([0, 1]) : \begin{aligned} & u^{(2n-i)}(t) \geq 0, t \in [0, 1], i \equiv 0 \pmod{4}, \\ & u^{(2n-i)}(1) \geq 0, i \equiv 1 \pmod{4}, \\ & u^{(2n-i)}(0) \geq 0, i \equiv 2 \pmod{4}, \\ & u^{(2n-i)}(0) \geq 0, i \equiv 3 \pmod{4}, \\ & \min_{t \in [a_{2n-4}, b_{2n-4}]} u^{(2n-4)}(t) \geq \xi_{2n-4}(a_{2n-4}, b_{2n-4}) \|u^{(2n-4)}\|_{[0,1]}, \\ & \min_{t \in [0, b_{2n-3}]} u^{(2n-3)}(t) \geq \xi_{2n-3}(b_{2n-3}) \|u^{(2n-3)}\|_{[0,1]} \end{aligned} \right\}.$$

Thus, for any nonlinearity f satisfying (H_6), (H_7) and conditions of either Theorem 6.3.1 or Theorem 6.4.3, it is possible to find nontrivial solutions of problem (6.6.1).

6.7. Particular Case: a Third Order Eigenvalue Three-point Boundary Value Problem

In this section we will study the existence of solutions of the third order nonlinear differential equation

$$-u^{(3)}(t) = \lambda f(t, u(t), u'(t), u''(t)), \quad t \in [0, 1], \quad (6.7.1)$$

with $\lambda > 0$ a real parameter and $f: [0, 1] \times \mathbb{R}^3 \rightarrow [0, \infty)$ a L^1 -Carathéodory function, coupled with the three point boundary value conditions

$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \quad (6.7.2)$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$ are given constants.

This type of third order three-point boundary value problems can be seen as a particular case of multipoint problems (as in [156]), nonlocal problems (see [78]), functional problems (as in [35]) or, as we will show in this section, integral equations. Therefore all the applications for the above type of problems hold for our problem. More precisely, these third order three-point boundary value problems arise in several areas of applied mathematics and physics, such as the deflection of a curved beam with a constant or varying cross section, three layer beams, electromagnetic waves, gravity driven flows, study of the equilibrium states of a heated bar, and other ones contained in [65].

A precedent problem

$$\begin{cases} u^{(3)}(t) + a(t)f(u(t)) = 0, & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \end{cases}$$

has been considered in [69]. There, the authors established some of the properties of the Green's function related to previous problem. From them, they built a suitable cone and applied Guo-Krasnoselskii's Theorem to assure the existence of a positive solution of the problem.

Recently, in [116], the authors considered the following system

$$\begin{cases} -u^{(3)}(t) = f(t, v(t), v'(t)), & t \in [0, 1], \\ -v^{(3)}(t) = f(t, u(t), u'(t)), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \\ v(0) = v'(0) = 0, & v'(1) = \alpha v'(\eta). \end{cases}$$

They studied the properties of the first derivative of the Green's function related to the problem and used them to construct a cone K such that there exist $u, v \in K$

which constitute a positive solution of the system. To do this, they also use Guo-Krasnoselskii's Theorem (see [68]).

A similar nonlinear fourth-order boundary value problem has been treated in [83], where the authors studied the existence of nonzero and positive solutions by means of monotone iterative techniques and lower and upper solutions.

In this section, we will study a generalization of previous equations by considering that the nonlinearity f depends on the solution and its first and second order derivatives. Because of this, we need to examine the properties of the second derivative of the Green's function.

We shall adapt the existence results given in previous sections to this particular case. Moreover, we will give some sufficient conditions for nonexistence of solution. All the results in this section are collected in [32].

6.7.1. Preliminary Results

In this subsection we compile several results regarding some properties of both the Green's function related to the problem and its first derivative. Next, we prove some inequalities satisfied by the second derivative of the Green's function.

The Green's function related to the homogeneous problem

$$\begin{cases} -u^{(3)}(t) = 0, & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \end{cases}$$

is given by the following expression (see [69])

$$G(t, s) = \frac{1}{2(1 - \alpha\eta)} \begin{cases} (2ts - s^2)(1 - \alpha\eta) + t^2s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t^2(1 - \alpha\eta) + t^2s(\alpha - 1), & t \leq s \leq \eta, \\ (2ts - s^2)(1 - \alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\ t^2(1 - s), & \max\{\eta, t\} \leq s. \end{cases}$$

Next lemmas establish some properties of the Green's function.

Lemma 6.7.1. ([69, Lemma 2.2]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$0 \leq G(t, s) \leq \phi_0(s) = \frac{1 + \alpha}{1 - \alpha\eta} s(1 - s), \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

Lemma 6.7.2. ([69, Lemma 2.3]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$G(t, s) \geq \xi_0 \phi_0(s), \quad \forall (t, s) \in \left[\frac{\eta}{\alpha}, \eta\right] \times [0, 1],$$

with $0 < \xi_0 = \frac{\eta^2}{2\alpha^2(1 + \alpha)} \min\{\alpha - 1, 1\} < 1$.

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In particular, with the notation introduced in Section 6.2, we are taking $h_0 \equiv \phi_0$, $[m_0, n_0] = [0, 1]$, $[c_0, d_0] = [0, 1]$ and $[a_0, b_0] = [\frac{\eta}{\alpha}, \eta]$.

The first derivative of G is given by (see [116])

$$\frac{\partial G}{\partial t}(t, s) = \frac{1}{(1 - \alpha\eta)} \begin{cases} s(1 - \alpha\eta) + ts(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t(1 - \alpha\eta) + ts(\alpha - 1), & t \leq s \leq \eta, \\ s(1 - \alpha\eta) + t(\alpha\eta - s), & \eta \leq s \leq t, \\ t(1 - s), & \max\{\eta, t\} \leq s, \end{cases}$$

and satisfies the following properties.

Lemma 6.7.3. ([116, Lemma 3]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$0 \leq \frac{\partial G}{\partial t}(t, s) \leq \phi_1(s) = \frac{1 - s}{1 - \alpha\eta}, \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

Lemma 6.7.4. ([116, Lemma 4]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$\frac{\partial G}{\partial t}(t, s) \geq \xi_1 \phi_1(s), \quad \forall (t, s) \in \left[\frac{\eta}{\alpha}, \eta\right] \times [0, 1],$$

with $0 < \xi_1 = \eta < 1$.

In particular, with the notation introduced in Section 6.2, we are taking $h_1 \equiv \phi_1$, $[m_1, n_1] = [0, 1]$, $[c_1, d_1] = [0, 1]$ and $[a_1, b_1] = [\frac{\eta}{\alpha}, \eta]$.

The second derivative of G is given by

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{1}{1 - \alpha\eta} \begin{cases} s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ 1 - \alpha\eta + s(\alpha - 1), & t < s \leq \eta, \\ \alpha\eta - s, & \eta \leq s < t, \\ 1 - s, & \max\{\eta, t\} \leq s. \end{cases}$$

It is immediate to verify that it satisfies the following conditions.

Lemma 6.7.5. Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$\frac{\partial^2 G}{\partial t^2}(t, s) \geq 0, \quad \forall (t, s) \in ([0, 1] \times [0, 1]) \setminus A,$$

where

$$A = \{(t, s) \in [0, 1] \times [0, 1]; \alpha\eta < t \leq 1, \alpha\eta < s < t\}.$$

Remark 6.7.6. Note that, in particular, $\frac{\partial^2 G}{\partial t^2}(t, s) \geq 0$ for all $(t, s) \in [0, \alpha\eta] \times [0, 1]$. This implies that, with the notation introduced in Section 6.2, $[m_2, n_2] = [0, \alpha\eta]$.

Lemma 6.7.7. Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,

$$\frac{\partial^2 G}{\partial t^2}(t, s) \leq \phi_2(s), \quad \forall (t, s) \in \left[\frac{\eta}{\alpha}, 1\right] \times [0, 1]$$

and

$$-1 \leq \frac{\partial^2 G}{\partial t^2}(t, s) \leq \max \left\{ \phi_2(s), \frac{1-\eta}{1-\alpha\eta} \right\}, \quad \forall (t, s) \in [0, 1] \times [0, 1],$$

with

$$\begin{aligned} \phi_2(s) &= \frac{\alpha(1-\alpha\eta) + \eta(\alpha-1)}{\eta(\alpha-1)} \frac{\partial^2 G}{\partial t^2}(\eta, s) \\ &= \frac{\alpha(1-\alpha\eta) + \eta(\alpha-1)}{\eta(\alpha-1)(1-\alpha\eta)} \begin{cases} s(\alpha-1), & 0 \leq s \leq \eta, \\ 1-s, & \eta \leq s \leq 1. \end{cases} \end{aligned}$$

Proof. First, we will prove that $\frac{\partial^2 G}{\partial t^2}(t, s) \leq \phi_2(s)$ for all $(t, s) \in \left[\frac{\eta}{\alpha}, 1\right] \times [0, 1]$.

For $s \leq \min\{\eta, t\}$ and $s \geq \max\{\eta, t\}$ we have that $\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{\partial^2 G}{\partial t^2}(\eta, s)$ and, since $\frac{\alpha(1-\alpha\eta)}{\eta(\alpha-1)} + 1 > 1$, it is obvious that $\frac{\partial^2 G}{\partial t^2}(t, s) \leq \phi_2(s)$.

For $t \leq s \leq \eta$, we have that

$$\begin{aligned} \frac{\partial^2 G}{\partial t^2}(t, s) &= \frac{1-\alpha\eta + s(\alpha-1)}{1-\alpha\eta} = \frac{(\alpha-1) \left(\frac{\eta\alpha(1-\alpha\eta)}{\alpha\eta(\alpha-1)} + s \right)}{1-\alpha\eta} \\ &\leq \frac{s(\alpha-1) \left(\frac{\alpha(1-\alpha\eta)}{\eta(\alpha-1)} + 1 \right)}{1-\alpha\eta} = \phi_2(s). \end{aligned}$$

Finally, for $\eta \leq s \leq t$,

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{\alpha\eta - s}{1-\alpha\eta} \leq \frac{1-s}{1-\alpha\eta} = \frac{\partial^2 G}{\partial t^2}(\eta, s) \leq \phi_2(s).$$

Now, we will prove that

$$\frac{\partial^2 G}{\partial t^2}(t, s) \geq -1 \quad \text{for all } (t, s) \in [0, 1] \times [0, 1].$$

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It is immediate to verify that $\frac{\partial^2 G}{\partial t^2}(t, s) \geq 0$ for $s \leq \min\{\eta, t\}$, $t \leq s \leq \eta$ and $\max\{\eta, t\} \leq s$. On the other hand, for $\eta \leq s \leq t$, we have that

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{\alpha \eta - s}{1 - \alpha \eta} \geq \frac{\alpha \eta - 1}{1 - \alpha \eta} = -1,$$

and so the result holds.

Finally, we will prove that

$$\frac{\partial^2 G}{\partial t^2}(t, s) \leq \max \left\{ \phi_2(s), \frac{1 - \eta}{1 - \alpha \eta} \right\} \quad \text{for all } (t, s) \in [0, 1] \times [0, 1].$$

Obviously, it is enough to prove the inequality for $(t, s) \in [0, \frac{\eta}{\alpha}] \times [0, 1]$.

For $s \leq t$ and $s \geq \eta$, we have just seen that $\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{\partial^2 G}{\partial t^2}(\eta, s) \leq \phi_2(s)$.

On the other hand, for $t \leq s \leq \eta$, the following inequality holds

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{1 - \alpha \eta + s(\alpha - 1)}{1 - \alpha \eta} \leq \frac{1 - \alpha \eta + \eta(\alpha - 1)}{1 - \alpha \eta} = \frac{1 - \eta}{1 - \alpha \eta}$$

and so the result is proved. □

Corollary 6.7.8. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then*

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| \leq h_2(s) = \max \left\{ \phi_2(s), \frac{1 - \eta}{1 - \alpha \eta} \right\}, \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

In addition, from Lemmas 6.7.5 and 6.7.7, we get the corollary below.

Corollary 6.7.9. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then*

$$0 \leq \frac{\partial^2 G}{\partial t^2}(t, s) \leq \phi_2(s), \quad \forall (t, s) \in \left[\frac{\eta}{\alpha}, \alpha \eta \right].$$

This implies that, with the notation introduced in Section 6.2, we are taking $[c_2, d_2] = \left[\frac{\eta}{\alpha}, \alpha \eta \right]$.

Remark 6.7.10. *We note that for any constant $c \in (0, 1)$ it would be possible to find a continuous function $g \neq \phi_2$ such that*

$$\frac{\partial^2 G}{\partial t^2}(t, s) \leq g(s), \quad \forall (t, s) \in [c, 1] \times [0, 1].$$

Lemma 6.7.11. *Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then,*

$$\frac{\partial^2 G}{\partial t^2}(t, s) \geq \frac{\partial^2 G}{\partial t^2}(\eta, s) \equiv \xi_2 \phi_2(s), \quad \forall (t, s) \in \left[\frac{\eta}{\alpha}, \eta\right] \times [0, 1],$$

with $0 < \xi_2 = \frac{\eta(\alpha-1)}{\alpha(1-\alpha\eta)+\eta(\alpha-1)}$.

Proof. For $s \leq t$ and $s \geq \eta$, we have that $\frac{\partial^2 G}{\partial t^2}(t, s) = \xi_2 \phi_2(s)$.

On the other hand, for $t \leq s \leq \eta$, it holds that

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \frac{1 - \alpha\eta + s(\alpha - 1)}{1 - \alpha\eta} \geq \frac{s(\alpha - 1)}{1 - \alpha\eta} = \xi_2 \phi_2(s).$$

□

Thus, with the notation of Section 6.2, we take $[a_2, b_2] = \left[\frac{\eta}{\alpha}, \eta\right]$.

Remark 6.7.12. *We note that for any interval $[a, b] \subset (0, \alpha\eta)$ it would be possible to find a constant ξ such that*

$$\frac{\partial^2 G}{\partial t^2}(t, s) \geq \xi \phi_2(s), \quad \forall (t, s) \in [a, b] \times [0, 1].$$

However, for the sake of simplicity, we have chosen $\left[\frac{\eta}{\alpha}, \eta\right]$ to maintain the same interval than in Lemmas 6.7.2 and 6.7.4.

Remark 6.7.13. *We point out that, on the contrary to function G and $\frac{\partial G}{\partial t}$, it is not possible to find a continuous function $\tilde{\phi}_2(s)$ such that*

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| \leq \tilde{\phi}_2(s), \quad \forall (t, s) \in [0, 1] \times [0, 1]$$

and

$$\frac{\partial^2 G}{\partial t^2}(t, s) \geq \tilde{\xi}_2 \tilde{\phi}_2(s), \quad \forall (t, s) \in [a, b] \times [0, 1],$$

with $[a, b] \subset [0, 1]$ and $\tilde{\xi}_2 \in (0, 1)$.

This is due to the fact that for $s \geq \alpha\eta$,

$$\left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| = \begin{cases} \frac{s - \alpha\eta}{1 - \alpha\eta}, & s \leq t, \\ \frac{1 - s}{1 - \alpha\eta}, & t \leq s. \end{cases}$$

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As a consequence, if there exists $\tilde{\phi}_2$ satisfying the previous conditions, it would necessarily satisfy that

$$\tilde{\phi}_2(s) \geq \max \left\{ \frac{s - \alpha \eta}{1 - \alpha \eta}, \frac{1 - s}{1 - \alpha \eta} \right\} = \begin{cases} \frac{1-s}{1-\alpha \eta}, & s \leq \frac{1+\alpha \eta}{2}, \\ \frac{s-\alpha \eta}{1-\alpha \eta}, & s \geq \frac{1+\alpha \eta}{2}, \end{cases}$$

for $s \geq \alpha \eta$, and so $\tilde{\phi}_2(1) \geq 1$.

On the other hand, we have that $\frac{\partial^2 G}{\partial t^2}(t, 1) = 0$ so, if there exists $\tilde{\phi}_2$ in the previous conditions, it would happen that

$$0 = \frac{\partial^2 G}{\partial t^2}(t, 1) \geq \tilde{\xi}_2 \tilde{\phi}_2(1) \geq \tilde{\xi}_2 > 0,$$

which is a contradiction.

6.7.2. Existence and Multiplicity of Solutions

Now, following the line developed in previous sections of this chapter, we will find solutions of problem (6.7.1)–(6.7.2) by identifying them with the fixed points of the integral operator

$$Tu(t) = \lambda \int_0^1 G(t, s) f(s, u(s), u'(s), u''(s)) \, ds, \quad t \in [0, 1]. \quad (6.7.3)$$

In particular, we need to verify that this problem satisfied all the required hypotheses, namely $(H_1) - (H_5)$ or, alternatively, $(H_1) - (H_4)$ and (\tilde{H}_5) :

- (H_1) This condition holds as a consequence of the general properties of the Green's function.
- (H_2) In this case $J_0 = J = \{0, 1, 2\}$. In particular, as we have just commented, we could take $[m_0, n_0] = [0, 1]$, $[m_1, n_1] = [0, 1]$ and $[m_2, n_2] = [0, \alpha \eta]$.
- (H_3) This hypothesis holds for $h_0 \equiv \phi_0$ given in Lemma 6.7.1, $h_1 \equiv \phi_1$ given in Lemma 6.7.3 and h_2 given in Corollary 6.7.8.
- (H_4) In this case $J_1 = J_0 = \{0, 1, 2\}$ and functions ϕ_0 , ϕ_1 and ϕ_2 are given in Lemmas 6.7.1, 6.7.3 and 6.7.7. In particular, as we have mentioned before, we may choose

- $[c_0, d_0] = [0, 1]$ and $[a_0, b_0] = [\frac{\eta}{\alpha}, \eta]$,
- $[c_1, d_1] = [0, 1]$ and $[a_1, b_1] = [\frac{\eta}{\alpha}, \eta]$,

$$\bullet [c_2, d_2] = \left[\frac{\eta}{\alpha}, \alpha \eta\right] \text{ and } [a_2, b_2] = \left[\frac{\eta}{\alpha}, \eta\right].$$

Moreover, it is clear that $\phi_0, \phi_1, \phi_2 \in L^1([0, 1])$ and

$$\int_{a_j}^{b_j} \phi_j(s) \, ds > 0 \text{ for } j = 0, 1, 2.$$

$(H_5), (\tilde{H}_5)$ Since $[c_0, d_0] = [0, 1]$, we may choose $i_0 = 0$.

Taking into account the properties satisfied by the Green's function and its derivatives, we will consider the cone

$$K = \left\{ u \in \mathcal{C}^2([0, 1], \mathbb{R}) : \begin{aligned} &u(t) \geq 0, \, t \in [0, 1], \, u'(t) \geq 0, \, t \in [0, 1], \\ &u''(t) \geq 0, \, t \in [0, \alpha \eta], \, \min_{t \in [\frac{\eta}{\alpha}, \eta]} u(t) \geq \xi_0 \|u\|_{\infty}, \\ &\min_{t \in [\frac{\eta}{\alpha}, \eta]} u'(t) \geq \xi_1 \|u'\|_{\infty}, \, \min_{t \in [\frac{\eta}{\alpha}, \eta]} u''(t) \geq \xi_2 \|u''\|_{[\frac{\eta}{\alpha}, \alpha \eta]} \end{aligned} \right\}$$

with ξ_0, ξ_1 and ξ_2 defined in previous subsection.

Remark 6.7.14. In [32], last condition in the definition of the cone K is

$$\min_{t \in [\frac{\eta}{\alpha}, \eta]} u''(t) \geq \xi_2 \|u''\|_{[\frac{\eta}{\alpha}, \eta]}.$$

However, as we have seen in this chapter, it is possible to take the interval $[c_2, d_2] = [\frac{\eta}{\alpha}, \alpha \eta]$ instead of $[\frac{\eta}{\alpha}, \eta]$. This way, we get a smaller cone, which makes the location of the possible solutions of the problem more precise.

Now, if the nonlinearity f satisfies hypotheses $(H_6), (H_7)$ and either conditions of Theorem 6.3.1 or 6.4.3, we could guarantee the existence of nontrivial solutions of problem (6.7.1)–(6.7.2).

Next, for the sake of completeness, we will give the exact expression of the constants $\Lambda_i, i = 1, \dots, 5$, involved in Theorem 6.3.1:

$$\blacksquare \Lambda_1 = \frac{\alpha + 1}{6(1 - \alpha \eta)},$$

$$\blacksquare \Lambda_2 = \frac{1}{2(1 - \alpha \eta)},$$

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$$\begin{aligned}
 \blacksquare \Lambda_3 &= \frac{\alpha^2 - 2\alpha(\alpha^2 + 1)\eta + (\alpha^4 + 3\alpha^3 + \alpha + 1)\eta^2}{2(\alpha - 1)\eta(\alpha\eta - 1)(\alpha((\alpha - 1)\eta - 1) + \eta)} \\
 &\quad + \frac{-2(\alpha(\alpha(\alpha(2\alpha - 3) + 5) - 3) + 1)\eta^3}{2(\alpha - 1)\eta(\alpha\eta - 1)(\alpha((\alpha - 1)\eta - 1) + \eta)} \\
 &\quad + \frac{\alpha^2(\alpha((\alpha - 2)\alpha + 3) - 1)\eta^4}{2(\alpha - 1)\eta(\alpha\eta - 1)(\alpha((\alpha - 1)\eta - 1) + \eta)}, \\
 \blacksquare \Lambda_4 &= \frac{\eta^4(\alpha^3(2\eta - 3) + 3\alpha - 2\eta)}{12\alpha^5(\alpha\eta - 1)} \min\{\alpha - 1, 1\}, \\
 \blacksquare \Lambda_5 &= \frac{(\alpha - 1)\eta^2(\alpha(\eta - 2) + \eta)}{2\alpha^2(\alpha\eta - 1)}.
 \end{aligned}$$

Also for the sake of completeness, we give the exact expression of the components involved in the formulas of $\frac{1}{N}$ and $\frac{1}{M_i}$, $i = 0, 1, 2$ which appear in Lemmas 6.4.1 and 6.4.2:

$$\int_0^1 G(t, s) \, ds = \frac{1}{12} t^2 \left(\frac{3(\alpha\eta^2 - 1)}{\alpha\eta - 1} - 2t \right)$$

and

$$\sup_{t \in [0, 1]} \int_0^1 G(t, s) \, ds = \int_0^1 G(1, s) \, ds = \frac{\alpha\eta(2 - 3\eta) + 1}{12(1 - \alpha\eta)}.$$

Moreover,

$$\int_0^1 \frac{\partial G}{\partial t}(t, s) \, ds = \frac{t(\alpha\eta(\eta - t) + t - 1)}{2(\alpha\eta - 1)}$$

and

$$\sup_{t \in [0, 1]} \int_0^1 \frac{\partial G}{\partial t}(t, s) \, ds = \int_0^1 \frac{\partial G}{\partial t}(1, s) \, ds = \frac{\alpha\eta(1 - \eta)}{2(1 - \alpha\eta)}.$$

Finally,

$$\int_0^1 \frac{\partial^2 G}{\partial t^2}(t, s) \, ds = \begin{cases} \frac{\alpha\eta(\eta - 2t) + 2t - 1}{2(\alpha\eta - 1)}, & t \leq \alpha\eta, \\ \frac{-2\alpha^2\eta^2 + \alpha\eta(\eta + 2t) - 2(t - 1)t - 1}{2(\alpha\eta - 1)}, & t > \alpha\eta, \end{cases}$$

and

$$\sup_{t \in [0, 1]} \int_0^1 \frac{\partial^2 G}{\partial t^2}(t, s) \, ds = \int_0^1 \frac{\partial^2 G}{\partial t^2}(0, s) \, ds = \frac{1 - \alpha\eta^2}{2(1 - \alpha\eta)}.$$

Now, it is easy to verify that

$$\frac{1}{N} = \max \left\{ \frac{\alpha \eta (2 - 3\eta) + 1}{12(1 - \alpha \eta)}, \frac{\alpha \eta (1 - \eta)}{2(1 - \alpha \eta)}, \frac{1 - \alpha \eta^2}{2(1 - \alpha \eta)} \right\} = \frac{1 - \alpha \eta^2}{2(1 - \alpha \eta)}.$$

On the other hand, for $t \in [\frac{\eta}{\alpha}, \eta]$,

$$\int_{\frac{\eta}{\alpha}}^{\eta} G(t, s) \, ds = \frac{1}{12} \left(\frac{2\eta^3}{\alpha^3} - \frac{6\eta^2 t}{\alpha^2} + \frac{3\eta t^2 (\alpha (\alpha (\alpha \eta + \eta - 2) + \eta) - \eta)}{\alpha^2 (\alpha \eta - 1)} - 2t^3 \right)$$

and

$$\frac{1}{M_0} = \inf_{t \in [\frac{\eta}{\alpha}, \eta]} \int_{\frac{\eta}{\alpha}}^{\eta} G(t, s) \, ds = \int_{\frac{\eta}{\alpha}}^{\eta} G\left(\frac{\eta}{\alpha}, s\right) \, ds = \frac{(\alpha - 1)\eta^3 (\alpha (2 - \alpha \eta) - \eta)}{4\alpha^4 (1 - \alpha \eta)}.$$

In addition,

$$\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial G}{\partial t}(t, s) \, ds = \frac{-\alpha^2 \eta t (\alpha t + 2) + \alpha^2 t^2 + \eta^2 ((\alpha^3 + \alpha^2 + \alpha - 1)t + 1) - \alpha \eta^3}{2\alpha^2 (\alpha \eta - 1)}$$

and

$$\begin{aligned} \frac{1}{M_1} &= \inf_{t \in [\frac{\eta}{\alpha}, \eta]} \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial G}{\partial t}(t, s) \, ds = \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial G}{\partial t}\left(\frac{\eta}{\alpha}, s\right) \, ds \\ &= \frac{(\alpha - 1)\eta^2 (\alpha (2 - \alpha \eta) - \eta)}{2\alpha^3 (1 - \alpha \eta)}. \end{aligned}$$

Finally,

$$\int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial t^2}(t, s) \, ds = \frac{\eta (\alpha (\alpha (\alpha \eta + \eta - 2) + \eta) - \eta)}{2\alpha^2 (\alpha \eta - 1)} - t$$

and

$$\frac{1}{M_2} = \inf_{t \in [\frac{\eta}{\alpha}, \eta]} \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial t^2}(t, s) \, ds = \int_{\frac{\eta}{\alpha}}^{\eta} \frac{\partial^2 G}{\partial t^2}(\eta, s) \, ds = \frac{(\alpha - 1)^2 (\alpha + 1) \eta^2}{2\alpha^2 (1 - \alpha \eta)}.$$

We will show now two examples in which we will be able to warrant the existence of nontrivial solutions of problem (6.7.1)-(6.7.2).

Example 6.7.15. *Let's consider the problem with*

$$f(t, x, y, z) = \frac{h(t)}{x^2 + y^2 + z^2},$$

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where $c_1 \geq h(t) \geq c_2 > 0$ for all $t \in [0, 1]$, and η and α arbitrarily chosen, that is,

$$\begin{cases} -u^{(3)}(t) = \lambda \frac{h(t)}{(u(t))^2 + (u'(t))^2 + (u''(t))^2}, & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta). \end{cases}$$

In this case,

$$\begin{aligned} f_0 &= \lim_{|x|, |y|, |z| \rightarrow 0} \min_{t \in [0, 1]} \frac{f(t, x, y, z)}{|x| + |y| + |z|} \\ &= \lim_{|x|, |y|, |z| \rightarrow 0} \frac{\min_{t \in [0, 1]} h(t)}{(x^2 + y^2 + z^2)(|x| + |y| + |z|)} = +\infty \end{aligned}$$

and

$$\begin{aligned} f^\infty &= \lim_{|x|, |y|, |z| \rightarrow \infty} \max_{t \in [0, 1]} \frac{f(t, x, y, z)}{|x| + |y| + |z|} \\ &= \lim_{|x|, |y|, |z| \rightarrow \infty} \frac{\max_{t \in [0, 1]} h(t)}{(x^2 + y^2 + z^2)(|x| + |y| + |z|)} = 0, \end{aligned}$$

so Theorem 6.3.1 assures that there exists at least a positive solution of the problem for all $\lambda > 0$.

On the other hand, let $\rho > 0$. Then,

$$f^\rho = \sup \left\{ \frac{h(t)}{\rho(x^2 + y^2 + z^2)}; (t, x, y, z) \in [0, 1] \times [0, \rho] \times [0, \rho] \times [-\rho, \rho] \right\} = \infty$$

so it is not possible to find a positive ρ such that $\lambda \frac{f^\rho}{m} < 1$ and, consequently, Theorem 6.4.3 can not be applied in this case.

Example 6.7.16. Let's consider the problem with

$$f(t, x, y, z) = h(t)(x^2 + y^2 + z^2 + 1),$$

where $c_1 \geq h(t) \geq c_2 > 0$ for all $t \in [0, 1]$, $\eta = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, that is,

$$\begin{cases} -u^{(3)}(t) = \lambda h(t)((u(t))^2 + (u'(t))^2 + (u''(t))^2 + 1), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \frac{3}{2} u'(\frac{1}{2}). \end{cases}$$

In this case,

$$\begin{aligned} f_0 &= \lim_{|x|, |y|, |z| \rightarrow 0} \min_{t \in [0, 1]} \frac{f(t, x, y, z)}{|x| + |y| + |z|} \\ &= \lim_{|x|, |y|, |z| \rightarrow 0} \frac{(\min_{t \in [0, 1]} h(t))(x^2 + y^2 + z^2 + 1)}{|x| + |y| + |z|} = \infty \end{aligned}$$

and

$$\begin{aligned} f^\infty &= \lim_{|x|,|y|,|z| \rightarrow \infty} \max_{t \in [0,1]} \frac{f(t, x, y, z)}{|x| + |y| + |z|} \\ &= \lim_{|x|,|y|,|z| \rightarrow \infty} \frac{(\max_{t \in [0,1]} h(t)) (x^2 + y^2 + z^2 + 1)}{|x| + |y| + |z|} = \infty, \end{aligned}$$

so Theorem 6.3.1 cannot be applied.

However, we will see that Theorem 6.4.3 lets us ensure the existence of at least one positive solution for certain values of λ .

Let $\rho_1, \rho_2 > 0$. Then,

$$f_{\rho_1}^0 = f_{\rho_1}^1 = f_{\rho_1}^2 = \frac{1}{\rho_1} \inf_{t \in [\frac{1}{3}, \frac{1}{2}]} h(t)$$

and

$$f^{\rho_2} = \frac{1 + 3\rho_2^2}{\rho_2} \sup_{t \in [0,1]} h(t).$$

Moreover, $\frac{1}{N} = \frac{5}{4}$ and $\frac{1}{M_1} = \max \left\{ \frac{1}{M_0}, \frac{1}{M_1}, \frac{1}{M_2} \right\} = \frac{11}{108}$. As a consequence of (C1) in Theorem 6.4.3, for any $\rho_1, \rho_2 > 0$ such that $\rho_1 < c\rho_2 = \frac{\rho_2}{90}$ and

$$\frac{108\rho_1}{11 \inf_{t \in [\frac{1}{3}, \frac{1}{2}]} h(t)} < \frac{4\rho_2}{5(1 + 3\rho_2^2) \sup_{t \in [0,1]} h(t)},$$

there exists at least a non trivial solution of problem (6.7.1)-(6.7.2) for all

$$\lambda \in \left(\frac{108\rho_1}{11 \inf_{t \in [\frac{1}{3}, \frac{1}{2}]} h(t)}, \frac{4\rho_2}{5(1 + 3\rho_2^2) \sup_{t \in [0,1]} h(t)} \right).$$

In particular, it can be deduced that there exists at least a non trivial solution of problem (6.7.1)-(6.7.2) for all

$$\lambda \in \left(0, \frac{2}{5\sqrt{3} \sup_{t \in [0,1]} h(t)} \right).$$

6.7.3. Results of Non Existence of Solution

We will finish this subsection with a result which gives some conditions to ensure that the integral equation (6.7.3) has not nontrivial solution in K .

Theorem 6.7.17. *If one of the following conditions hold*

- (i) $f(t, x, y, z) < \tilde{m} \max\{x, y, |z|\}$ for every $t \in [0, 1]$, $x, y \geq 0$ and $z \in \mathbb{R}$, where

$$\frac{1}{\tilde{m}} = \max \left\{ \sup_{t \in [0, 1]} \lambda \int_0^1 G(t, s) \, ds, \sup_{t \in [0, 1]} \lambda \int_0^1 \frac{\partial G}{\partial t}(t, s) \, ds, \sup_{t \in [0, 1]} \lambda \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t, s) \right| \, ds \right\},$$

- (ii) $f(t, x, y, z) > Mx$ for every $t \in [a, b] \subset [\frac{\eta}{\alpha}, \eta]$, with $a \neq b$, $x, y \geq 0$ and $z \in \mathbb{R}$, where

$$\frac{1}{M} = \inf_{t \in [a, b]} \lambda \int_a^b G(t, s) \, ds,$$

- (iii) $f(t, x, y, z) > \tilde{M}y$ for every $t \in [a, b] \subset [\frac{\eta}{\alpha}, \eta]$, with $a \neq b$, $x, y \geq 0$ and $z \in \mathbb{R}$, where

$$\frac{1}{\tilde{M}} = \inf_{t \in [a, b]} \lambda \int_a^b \frac{\partial G}{\partial t}(t, s) \, ds,$$

then problem (6.7.1)–(6.7.2) has not nontrivial solution in K .

Proof. We will only prove (i) and (ii) since item (iii) is totally analogous to (ii).

- (i) Suppose, on the contrary, that there exists $u \in K$ such that $u = Tu$. Let $t_0 \in [0, 1]$ be such that $\|u\|_\infty = u(t_0)$. Then,

$$\begin{aligned} \|u\|_\infty &= \lambda \int_0^1 G(t_0, s) f(s, u(s), u'(s), u''(s)) \, ds \\ &< \lambda \int_0^1 G(t_0, s) \tilde{m} \max\{u(s), u'(s), |u''(s)|\} \, ds \\ &\leq \lambda \tilde{m} \|u\| \int_0^1 G(t_0, s) \, ds \leq \|u\|. \end{aligned}$$

Now, let $t_1 \in [0, 1]$ such that $\|u'\|_\infty = u'(t_1)$. Then,

$$\begin{aligned} \|u'\|_\infty &= \lambda \int_0^1 \frac{\partial G}{\partial t}(t_1, s) f(s, u(s), u'(s), u''(s)) \, ds \\ &< \lambda \|u\| \int_0^1 \frac{\partial G}{\partial t}(t_1, s) \tilde{m} \max\{u(s), u'(s), |u''(s)|\} \, ds \\ &\leq \lambda \tilde{m} \|u\| \int_0^1 \frac{\partial G}{\partial t}(t_1, s) \, ds \leq \|u\|. \end{aligned}$$

Finally, let $t_2 \in [0, 1]$ such that $\|u''\|_\infty = |u''(t_2)|$. Then,

$$\begin{aligned} \|u''\|_\infty &= \left| \lambda \int_0^1 \frac{\partial^2 G}{\partial t^2}(t_2, s) f(s, u(s), u'(s), u''(s)) \, ds \right| \\ &\leq \lambda \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t_2, s) \right| |f(s, u(s), u'(s), u''(s))| \, ds \\ &< \lambda \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t_2, s) \right| \tilde{m} \max \{u(s), u'(s), |u''(s)|\} \, ds \\ &\leq \lambda \tilde{m} \|u\| \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t_2, s) \right| \, ds \leq \|u\|. \end{aligned}$$

Consequently, we obtain

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\} < \|u\|,$$

which is a contradiction.

- (ii) Suppose, on the contrary, that there exists $u \in K$ such that $u = Tu$. Let $t_0 \in [0, 1]$ be such that $u(t_0) = \min_{t \in [a, b]} u(t)$. Then, for $t \in [a, b]$ we have that

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t, s) f(s, u(s), u'(s), u''(s)) \, ds \\ &\geq \lambda \int_a^b G(t, s) f(s, u(s), u'(s), u''(s)) \, ds \\ &> M \lambda \int_a^b G(t, s) u(s) \, ds. \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} u(t_0) &= \min_{t \in [a, b]} u(t) > M \inf_{t \in [a, b]} \lambda \int_a^b G(t, s) u(s) \, ds \\ &\geq M u(t_0) \inf_{t \in [a, b]} \lambda \int_a^b G(t, s) \, ds = u(t_0), \end{aligned}$$

which is a contradiction.

□

Part II

Unbounded Domains



Part II is devoted to the study of nonlinear problems defined on unbounded domains.

Boundary value problems on unbounded intervals arise in many models of applied mathematics, such as in combustion theory, plasma physics, models of unsteady flow of a gas through semi-infinite porous media, to study the electrical potential of an isolated neutral atom... For more details, techniques and applications in this field we refer, for example, to [66, 67, 84, 93, 94, 99, 154], and the monograph [2].

There are many results in the recent literature in which the authors deal with differential or integral problems defined on unbounded intervals (see, for instance, [48, 54, 114, 117, 118] and the references therein). The main difficulties which appear while dealing with this kind of problems arise as a consequence of the lack of compactness of the domain. In particular, this makes it impossible to apply Ascoli-Arzelá's Theorem to prove the compactness of the operator. In all of the cited references the authors solve this problem by means of the following relatively compactness criterion (see [41, 121]) which involves some stability condition at $\pm\infty$:

Theorem 1 ([41, Section 2.12]). *Let E be a Banach space and $\mathcal{C}(\mathbb{R}, E)$ the space of all bounded continuous functions $x: \mathbb{R} \rightarrow E$. For a set $D \subset \mathcal{C}(\mathbb{R}, E)$ to be relatively compact, it is necessary and sufficient that:*

1. *D is uniformly bounded.*
2. *Functions from D are equicontinuous on every compact subinterval of $[0, \infty)$.*
3. *Functions from D are equiconvergent at $+\infty$, that is, given $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geq T$, we have that*

$$\left\| x(t) - \lim_{t \rightarrow \infty} x(t) \right\| < \varepsilon.$$

By using the previous result, the authors of the aforementioned references prove the existence of solutions of differential or integral problems by means of either Schauder's fixed point Theorem (Theorem 1.2.3) or lower and upper solutions method.

In this Thesis we will show three different approaches to prove the existence of solutions of both differential and integral problems on unbounded domains.

First, in Chapter 7, we will study a nonlinear resonant problem. The difficulties in this kind of problems arise as a consequence of the noninvertibility of the differential operator, which makes it impossible to transform it into an integral operator. We will solve this problem by constructing a modified problem (whose solutions will be solutions of the original one) which will be nonresonant. Then, we will define an integral operator, we will prove its compactness by means of Theorem 1 and we will

use the lower and upper solutions method to prove the existence of fixed points of the aforementioned integral operator. All the results in this chapter are collected in [103].

In Chapter 8 we will solve the problem of the lack of compactness of the domain by defining a new Banach space which will let us use Ascoli-Arzelá's Theorem. Moreover, this Banach space will include some asymptotic conditions which will make it possible to predict not only the existence of solutions of the considered problems but also its asymptotic behavior. In this chapter we present two different methods to prove the existence of fixed points of integral operator: the first one based on the fixed point index properties and the second one on spectral theory. The results in this chapter can be found in [34] and [33].

Finally, Chapter 9 presents a problem which needs to be solved using different techniques. In particular, we will consider an initial value problem with a ϕ -Laplacian. This problem will be singular, which makes it impossible to transform it into an equivalent integral problem with a Green's function as kernel. Therefore, we will need to consider a different approach in order to prove the existence of solutions. In particular, we will focus our attention on unbounded solutions. Results in this chapter are included in [131].



On Multi-point Resonant Problems on the Half-line

In this chapter, we will deal with a nonlinear resonant problem defined on the half-line.

Resonant problems have been studied for many years under a huge variety of techniques: degree theory (see, for instance, [6, 53, 88, 124]), Lyapunov–Schmidt arguments (see [105]), a Leggett–Williams theorem (see [58, 119]), fixed point and fixed point index theories (see [5, 72, 79, 155]) or monotone methods together with lower and upper solutions techniques (see [3]), among others.

The main problem when dealing with resonant problems is that the related Green's function does not exist. This makes it impossible to construct an equivalent integral problem in the same way than in previous chapters. This issue is overcome applying several techniques.

From a theoretical point of view, resonant problems can be formulated as an equation $Lu = Nu$, where L is a noninvertible operator. Once the problem is formulated in such a way, if operators L and N satisfy certain conditions, the existence of a solution can be ensured (see, for instance, [48, 85]).

Our approach will be different: we will construct a modified problem (which will be shown to be equivalent to the original one) which will be nonresonant. Thus, this modified problem will have a related Green's function, so we will be able to transform it into an equivalent integral problem. The fixed points of this integral operator will be solutions of the aforementioned modified problem and, consequently, of the original one. This technique is also applied in [23, 36].

Our construction of the modified problem will have a second important advantage in order to find solutions of the nonlinear problem. What we will do is to construct a problem whose related Green's function belongs to $L^1[0, \infty) \cap L^\infty[0, \infty)$ and this will make it possible to ensure the compactness of the integral operator when the nonlinearity satisfies either L^1 or L^∞ -Carathéodory conditions. This way, we are able to solve nonlinear problems defined on unbounded domains in which the nonlinearity satisfies weaker properties than the ones usually required in the literature.

In this sense, we would also like to mention that this technique of modifying the problem in order to obtain another one whose related Green's function belongs to

$L^1[0, \infty) \cap L^\infty[0, \infty)$, is also applicable to problems without resonance. Thus, if we used this idea in problems defined on unbounded domains like

$$\begin{cases} u^{(4)}(t) + k u(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in \mathbb{R}, \\ u(\pm\infty) = 0, & u'(\pm\infty) = 0, \end{cases}$$

considered in [118], we could prove the existence of solutions of this problem in case that the nonlinearity satisfies L^∞ -Carathéodory conditions instead of L^1 -Carathéodory ones. The same could be said about, for instance, the problem considered in [114].

All the results in this chapter are collected in [103].

7.1. Introduction

We will prove the existence of bounded solutions for the multi-point boundary value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in [0, \infty), \\ u(0) = 0, & u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases} \quad (7.1.1)$$

where $\alpha_i > 0$ and $0 = \xi_1 < \dots < \xi_{m-1} < +\infty$. We will assume that the coefficients α_i satisfy the following resonant condition

$$\sum_{i=1}^{m-1} \alpha_i = 1. \quad (7.1.2)$$

It is easy to check that, under condition (7.1.2), the homogeneous boundary value problem related to (7.1.1),

$$\begin{cases} u''(t) = 0, & t \in [0, \infty), \\ u(0) = 0, & u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases}$$

has a nontrivial solution (in fact, every constant function is a solution of previous problem), that is, (7.1.1) is a resonant problem.

In [85] the authors studied the problem

$$\begin{cases} u''(t) + f(t, u(t)) = 0, & t \in [0, \infty), \\ u(0) = 0, & u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases} \quad (7.1.3)$$

also under condition (7.1.2). To deal with the resonant problem, they defined some suitable operators L and N such that the solutions of (7.1.3) coincide with the solutions of the equation $Lu = Nu$. We must point out that operator L defined in this reference is noninvertible. With this technique, they were able to find a solution in the space

$$E = \left\{ u \in \mathcal{C}[0, \infty), u(0) = 0, \sup_{t \in [0, \infty)} \frac{|u(t)|}{1+t} < +\infty \right\},$$

so clearly that solution could be unbounded.

Our arguments apply a different technique to find bounded solutions for problem (7.1.1). Moreover, we note that, on the contrary to [85], we allow the nonlinearity f to depend on the first derivative of u .

In [48], a similar third order boundary value problem is considered, namely

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)) = 0, & t \in [0, \infty), \\ u(0) = u'(0) = 0, u''(+\infty) = \sum_{i=1}^{m-2} \alpha_i u''(\xi_i), \end{cases}$$

coupled with the resonant condition

$$\sum_{i=1}^{m-2} \alpha_i = 1.$$

The techniques used in [48] are basically the same than in [85] and, again, the authors are able to find a solution which may be unbounded. On the other hand, they allow the nonlinearity f to depend on all the derivatives up to the highest possible order but, to do that, they ask for the following quite restrictive condition on the nonlinearity:

(H_0) $f : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is s^2 -Carathéodory, that is,

- (i) $f(\cdot, u, v, w)$ is measurable for each (u, v, w) fixed.
- (ii) $f(t, \cdot, \cdot, \cdot)$ is continuous for a. e. $t \in [0, \infty)$.
- (iii) For each $r > 0$ there exists $\psi_r \in L^1[0, \infty)$ with $t \psi_r, t^2 \psi_r \in L^1[0, \infty)$ such that

$$|f(t, u, v, w)| \leq \psi_r(t),$$

for all $(u, v, w) \in (-r, r) \times (-r, r) \times (-r, r)$ and a. e. $t \in [0, \infty)$.

In this chapter, we will look for solutions of problem (7.1.1) in the Banach space

$$X = \left\{ u \in C^1[0, \infty) : \sup_{t \in [0, \infty)} |u(t)| < \infty, \sup_{t \in [0, \infty)} |u'(t)| < \infty \right\}$$

equipped with the norm

$$\|u\| = \max \{ \|u\|_\infty, \|u'\|_\infty \},$$

where

$$\|v\|_\infty = \sup_{t \in [0, \infty)} |v(t)|.$$

In order to deal with the lack of compactness of the set X , we will use Theorem 1.

Moreover, we will assume that at least one of the two following conditions holds:

(H_1) The nonlinearity $f : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies L^1 -Carathéodory condition, that is,

- (i) $f(\cdot, u, v)$ is measurable for each (u, v) fixed.
- (ii) $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, \infty)$.
- (iii) For each $r > 0$ there exists $\varphi_r \in L^1[0, \infty)$ such that

$$|f(t, u, v)| \leq \varphi_r(t), \quad \forall (u, v) \in (-r, r) \times (-r, r), \text{ a.e. } t \in [0, \infty).$$

(H_2) The nonlinearity $f : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies L^∞ -Carathéodory condition, that is,

- (i) $f(\cdot, u, v)$ is measurable for each (u, v) fixed.
- (ii) $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, \infty)$.
- (iii) For each $r > 0$ there exists $\phi_r \in L^\infty[0, \infty)$ such that

$$|f(t, u, v)| \leq \phi_r(t), \quad \forall (u, v) \in (-r, r) \times (-r, r), \text{ a.e. } t \in [0, \infty).$$

We must point out that, although in this chapter we work with the second order problem, the same techniques could be applied to the third order problem. In this sense, we allow the nonlinearity f to depend on all the derivatives up to the highest possible order but using either hypothesis (H_1) or (H_2), instead of (H_0). This way, our hypotheses are clearly much less restrictive than (H_0) so our method improves the results in [48].

Finally, to prove the existence of solutions we will consider two different results. First of all we will use the very well-known Schauder's fixed point Theorem (Theorem 1.2.3).

On the other hand, we will also give a result to prove the existence of solutions based on the lower and upper solutions technique. To do that we need to introduce the following definition:

Definition 7.1.1. A function $\alpha \in X$ is said to be a lower solution of problem (7.1.1) if

$$\begin{cases} \alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), & t \in [0, \infty), \\ \alpha(0) \leq 0, \alpha'(+\infty) \geq \sum_{i=1}^{m-1} \alpha_i \alpha'(\xi_i). \end{cases}$$

A function $\beta \in X$ is said to be an upper solution of (7.1.1) if the reversed inequalities hold.

This chapter is divided into several sections: In Section 7.2, we construct an auxiliary differential problem whose solutions are the same than those of problem (7.1.1). In Section 7.3, this auxiliary problem is transformed into an integral one, for which some bounded solutions are found. These solutions are showed to be solutions of the original problem. Finally, Section 7.4 includes an example for which the results in [85] can not be applied.

7.2. Construction of the Auxiliary Problem

We will construct now a modified problem, which will be equivalent to (7.1.1), for which it is possible to construct the related Green's function.

Indeed, consider the modified problem

$$\begin{cases} u''(t) + k u'(t) + M u(t) = 0, & t \in [0, \infty), \\ u(0) = 0, u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases} \quad (7.2.1)$$

where k and M are positive constants such that $k^2 - 4M < 0$ and

$$\sum_{i=1}^{m-1} \alpha_i e^{-\frac{k \xi_i}{2}} \left(-\frac{k}{2} \sin(\gamma \xi_i) + \gamma \cos(\gamma \xi_i) \right) \neq 0,$$

with $\gamma = \sqrt{4M - k^2}$.

After detailed calculations, we obtain the explicit expression of the Green's function related to problem (7.2.1):

$$G(t, s) = \frac{e^{-\frac{k(t+s)}{2}}}{\gamma} \begin{cases} -\sin(\gamma t) h_l(s), & 0 \leq t \leq s, \quad \xi_{l-1} \leq s < \xi_l, \\ 0, & 0 \leq t \leq s, \quad \xi_{m-1} \leq s, \\ -\sin(\gamma t) h_l(s) + \sin(\gamma(s-t)), & 0 \leq s < t, \quad \xi_{l-1} \leq s < \xi_l, \\ \sin(\gamma(s-t)), & 0 \leq s < t, \quad \xi_{m-1} \leq s, \end{cases}$$

where

$$h_l(s) = \frac{\sum_{i=l}^{m-1} \alpha_i e^{-\frac{k\xi_i}{2}} \left(-\frac{k}{2} \sin(\gamma(s-\xi_i)) + \gamma \cos(\gamma(s-\xi_i)) \right)}{\sum_{i=1}^{m-1} \alpha_i e^{-\frac{k\xi_i}{2}} \left(-\frac{k}{2} \sin(\gamma\xi_i) + \gamma \cos(\gamma\xi_i) \right)}.$$

The first derivative of the Green's function is given by

$$\frac{\partial G}{\partial t}(t, s) = \frac{e^{-\frac{k(t+s)}{2}}}{\gamma} \begin{cases} \left(\frac{k}{2} \sin(\gamma t) - \gamma \cos(\gamma t) \right) h_l(s), & 0 \leq t \leq s, \quad \xi_{l-1} \leq s < \xi_l, \\ 0, & 0 \leq t \leq s, \quad \xi_{m-1} \leq s, \\ \left(\frac{k}{2} \sin(\gamma t) - \gamma \cos(\gamma t) \right) h_l(s) \\ - \frac{k}{2} \sin(\gamma(s-t)) - \cos(\gamma(s-t)), & 0 \leq s < t, \quad \xi_{l-1} \leq s < \xi_l, \\ - \frac{k}{2} \sin(\gamma(s-t)) - \cos(\gamma(s-t)), & 0 \leq s < t, \quad \xi_{m-1} \leq s. \end{cases}$$

Remark 7.2.1. It is easy to see that there exist two positive constants, C_1 and C_2 , such that

$$|G(t, s)| \leq C_1 e^{-\frac{k(t+s)}{2}}$$

and

$$\left| \frac{\partial G}{\partial t}(t, s) \right| \leq C_2 e^{-\frac{k(t+s)}{2}},$$

for all $(t, s) \in [0, \infty) \times [0, \infty)$.

As a consequence, it is clear that both $G(t, \cdot)$ and $\frac{\partial G}{\partial t}(t, \cdot)$ belong to the space $L^1[0, \infty) \cap L^\infty[0, \infty)$ for all $t \in [0, \infty)$.

7.3. Main Results

Consider now the following integral operator $T: X \rightarrow X$ defined by

$$Tu(t) = \int_0^\infty G(t, s) \left(f(s, u(s), u'(s)) + k u'(s) + M u(s) \right) ds. \quad (7.3.1)$$

It is clear that solutions of problem (7.1.1) are fixed points of operator T .

We will prove now that operator T is compact.

Lemma 7.3.1. *Assume that either (H_1) or (H_2) holds. Then operator T defined in (7.3.1) is compact.*

Proof. The proof will be divided into several steps.

Step 1: T is well-defined in X .

Given an arbitrary $u \in X$, we will prove that $Tu \in X$.

We will omit the proof that $Tu \in C^1[0, \infty)$ as it can be deduced from the proof of equicontinuity that we will present in Step 3.

We will see then that both $|Tu|$ and $|(Tu)'|$ are bounded on $[0, \infty)$. First, we will make the proof in case that hypothesis (H_1) holds. If $u \in X$, then there exists some $r > 0$ such that $\|u\| < r$. Therefore, for all $t \in [0, \infty)$, it holds that

$$\begin{aligned}
 |Tu(t)| &= \left| \int_0^\infty G(t, s) (f(s, u(s), u'(s)) + k u'(s) + M u(s)) \, ds \right| \\
 &\leq \int_0^\infty |G(t, s)| (|f(s, u(s), u'(s))| + k |u'(s)| + M |u(s)|) \, ds \\
 &\leq \int_0^\infty |G(t, s)| (\varphi_r(s) + (k + M) r) \, ds \\
 &\leq \int_0^\infty C_1 e^{-\frac{k(t+s)}{2}} (\varphi_r(s) + (k + M) r) \, ds \\
 &= C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \frac{2}{k} (k + M) r \right) \\
 &= C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right)
 \end{aligned} \tag{7.3.2}$$

and, analogously, for $t \in [0, \infty)$,

$$\begin{aligned}
 |(Tu)'(t)| &= \left| \int_0^\infty \frac{\partial G}{\partial t}(t, s) (f(s, u(s), u'(s)) + k u'(s) + M u(s)) \, ds \right| \\
 &\leq \int_0^\infty \left| \frac{\partial G}{\partial t}(t, s) \right| (|f(s, u(s), u'(s))| + k |u'(s)| + M |u(s)|) \, ds \\
 &\leq \int_0^\infty \left| \frac{\partial G}{\partial t}(t, s) \right| (\varphi_r(s) + (k + M) r) \, ds \\
 &\leq \int_0^\infty C_2 e^{-\frac{k(t+s)}{2}} (\varphi_r(s) + (k + M) r) \, ds \\
 &= C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right).
 \end{aligned} \tag{7.3.3}$$

Now, since $\varphi_r \in L^1[0, \infty)$ and $e^{-\frac{ks}{2}} \in L^\infty[0, \infty)$, it holds that the product $\varphi_r(s) e^{-\frac{ks}{2}} \in L^1[0, \infty)$. Thus, it is clear that

$$\sup_{t \in [0, \infty)} |Tu(t)| < \infty \quad \text{and} \quad \sup_{t \in [0, \infty)} |(Tu)'(t)| < \infty,$$

that is, $Tu \in X$.

On the other hand, if (H_2) holds instead of (H_1) , following similar steps to the previous case, we obtain the following upper bounds for $|Tu(t)|$ and $|(Tu)'(t)|$:

$$|Tu(t)| \leq C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \phi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right)$$

and

$$|(Tu)'(t)| \leq C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \phi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right).$$

In this case $\phi_r \in L^\infty[0, \infty)$ and, since $e^{-\frac{ks}{2}} \in L^1[0, \infty)$, we obtain that the product $\phi_r(s) e^{-\frac{ks}{2}} \in L^1[0, \infty)$. Therefore we conclude again that $Tu \in X$.

Step 2: T is a continuous operator.

We will detail the proof for the case in which (H_1) holds. For (H_2) the proof will be analogous, with the obvious changes, as it occurred in Step 1.

Consider the sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ and assume that it converges to u in X , that is,

$$\lim_{n \rightarrow \infty} u_n(t) = u(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} u'_n(t) = u'(t),$$

uniformly on t . Then, since $f(t, \cdot, \cdot)$ is continuous for a. e. $t \in [0, \infty)$, it is deduced that

$$\lim_{n \rightarrow \infty} f(s, u_n(s), u'_n(s)) = f(s, u(s), u'(s)) \quad \text{for a. e. } s \in [0, \infty).$$

Let's see that $\{Tu_n\}_{n \in \mathbb{N}}$ converges to Tu in X .

Since $\{u_n\}_{n \in \mathbb{N}}$ is convergent in X , there exists some $r > 0$ such that $\|u_n\| < r$

for all $n \in \mathbb{N}$. Now, if (H_1) holds, we have for all $t \geq 0$ that

$$\begin{aligned}
 |Tu_n(t) - Tu(t)| &\leq \int_0^\infty |G(t, s)| |f(s, u_n(s), u'_n(s)) - f(s, u(s), u'(s))| \, ds \\
 &\quad + \int_0^\infty |G(t, s)| (k |u'_n(s) - u'(s)| + M |u_n(s) - u(s)|) \, ds \\
 &\leq C_1 \int_0^\infty e^{-\frac{ks}{2}} |f(s, u_n(s), u'_n(s)) - f(s, u(s), u'(s))| \, ds \\
 &\quad + C_1 \int_0^\infty e^{-\frac{ks}{2}} (k |u'_n(s) - u'(s)| + M |u_n(s) - u(s)|) \, ds \\
 &\leq C_1 \int_0^\infty e^{-\frac{ks}{2}} (2\varphi_r(s) + 2(k + M)r) \, ds < \infty.
 \end{aligned}$$

Then, we deduce from Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \|Tu_n - Tu\|_\infty \\
 &\leq \lim_{n \rightarrow \infty} C_1 \int_0^\infty e^{-\frac{ks}{2}} |f(s, u_n(s), u'_n(s)) - f(s, u(s), u'(s))| \, ds \\
 &\quad + \lim_{n \rightarrow \infty} C_1 \int_0^\infty e^{-\frac{ks}{2}} (k |u'_n(s) - u'(s)| + M |u_n(s) - u(s)|) \, ds \\
 &= C_1 \int_0^\infty \lim_{n \rightarrow \infty} e^{-\frac{ks}{2}} |f(s, u_n(s), u'_n(s)) - f(s, u(s), u'(s))| \, ds \\
 &\quad + C_1 \int_0^\infty \lim_{n \rightarrow \infty} e^{-\frac{ks}{2}} (k |u'_n(s) - u'(s)| + M |u_n(s) - u(s)|) \, ds = 0.
 \end{aligned}$$

Analogously, we get that

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \|(Tu_n)' - (Tu)'\|_\infty \\
 &\leq C_2 \int_0^\infty \lim_{n \rightarrow \infty} e^{-\frac{ks}{2}} |f(s, u_n(s), u'_n(s)) - f(s, u(s), u'(s))| \, ds \\
 &\quad + C_2 \int_0^\infty \lim_{n \rightarrow \infty} e^{-\frac{ks}{2}} (k |u'_n(s) - u'(s)| + M |u_n(s) - u(s)|) \, ds = 0.
 \end{aligned}$$

Thus, $\{Tu_n\}_{n \in \mathbb{N}}$ converges to Tu in X .

Step 3: T is compact.

Again, we will make the proof only for the case in which (H_1) holds, being the one with (H_2) analogous.

Let B be a bounded subset of X , that is, there exists some $r > 0$ such that $\|u\| < r$, for all $u \in B$. Let us see that $T(B)$ is relatively compact in X .

(i) $T(B)$ is uniformly bounded:

If $u \in B$, then, for $t \in [0, \infty)$,

$$\begin{aligned} |Tu(t)| &\leq C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) \\ &\leq C_1 \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) =: M_1 > 0 \end{aligned}$$

and

$$\begin{aligned} |(Tu)'(t)| &\leq C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) \\ &\leq C_2 \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) =: M_2 > 0. \end{aligned}$$

Thus,

$$\|Tu\| \leq \max\{M_1, M_2\},$$

for all $u \in B$, that is, $T(B)$ is uniformly bounded.

(ii) $T(B)$ is equicontinuous:

We will see that functions in $T(B)$ are equicontinuous on $[0, \infty)$. Indeed, let $t_1, t_2 \in [0, \infty)$ and assume that $t_1 > t_2$. Then,

$$\begin{aligned} &|Tu(t_1) - Tu(t_2)| \\ &\leq \int_0^\infty |G(t_1, s) - G(t_2, s)| (|f(s, u(s), u'(s))| + k|u'(s)| + M|u(s)|) \, ds \\ &\leq \int_0^\infty |G(t_1, s) - G(t_2, s)| (\varphi_r(s) + (k + M)r) \, ds \\ &= \int_0^{t_2} |G(t_1, s) - G(t_2, s)| (\varphi_r(s) + (k + M)r) \, ds \\ &\quad + \int_{t_2}^{t_1} |G(t_1, s) - G(t_2, s)| (\varphi_r(s) + (k + M)r) \, ds \\ &\quad + \int_{t_1}^\infty |G(t_1, s) - G(t_2, s)| (\varphi_r(s) + (k + M)r) \, ds. \end{aligned} \tag{7.3.4}$$

We will find some suitable upper bounds for the difference $|G(t_1, s) - G(t_2, s)|$.

We will distinguish between three different cases.

1) For $0 \leq t_2 < t_1 \leq s$, we have two possibilities:

- If $\xi_{l-1} \leq s < \xi_l$ for some $2 \leq l \leq m-1$, then

$$\begin{aligned} & |G(t_1, s) - G(t_2, s)| \\ &= \frac{1}{\gamma} |h_l(s)| e^{-\frac{k s}{2}} \left| -e^{-\frac{k t_1}{2}} \sin(\gamma t_1) + e^{-\frac{k t_2}{2}} \sin(\gamma t_2) \right|. \end{aligned}$$

- If $s \geq \xi_{m-1}$, then

$$|G(t_1, s) - G(t_2, s)| = 0.$$

Moreover, we note that h_l is uniformly bounded, that is, there exists some positive constant C such that

$$|h_l(s)| \leq C, \quad \text{for all } s \in [0, \infty), \quad 2 \leq l \leq m-1.$$

Therefore, we can affirm that, for a given $\varepsilon > 0$, there exists some $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then, for $s \in (t_1, \infty)$, it holds that

$$|G(t_1, s) - G(t_2, s)| \leq \varepsilon e^{-\frac{k s}{2}}.$$

This implies that the third term of the last part of inequality (7.3.4) tends to zero with independence of the function $u \in B$.

2) Similarly, for $0 \leq s \leq t_2 < t_1$:

- If $\xi_{l-1} \leq s < \xi_l$ for some $2 \leq l \leq m-1$, then

$$\begin{aligned} & |G(t_1, s) - G(t_2, s)| \\ &\leq \frac{1}{\gamma} |h_l(s)| e^{-\frac{k s}{2}} \left| -e^{-\frac{k t_1}{2}} \sin(\gamma t_1) + e^{-\frac{k t_2}{2}} \sin(\gamma t_2) \right| \\ &\quad + \frac{1}{\gamma} e^{-\frac{k s}{2}} \left| e^{-\frac{k t_1}{2}} \sin(\gamma(s - t_1)) - e^{-\frac{k t_2}{2}} \sin(\gamma(s - t_2)) \right|. \end{aligned}$$

Last term in the previous sum can be upperly bounded as follows

$$\begin{aligned} & \left| e^{-\frac{k t_1}{2}} \sin(\gamma(s - t_1)) - e^{-\frac{k t_2}{2}} \sin(\gamma(s - t_2)) \right| \\ &\leq |\sin(\gamma s)| \left| e^{-\frac{k t_1}{2}} \cos(\gamma t_1) - e^{-\frac{k t_2}{2}} \cos(\gamma t_2) \right| \\ &\quad + |\cos(\gamma s)| \left| -e^{-\frac{k t_1}{2}} \sin(\gamma t_1) + e^{-\frac{k t_2}{2}} \sin(\gamma t_2) \right| \\ &\leq \left| e^{-\frac{k t_1}{2}} \cos(\gamma t_1) - e^{-\frac{k t_2}{2}} \cos(\gamma t_2) \right| \\ &\quad + \left| -e^{-\frac{k t_1}{2}} \sin(\gamma t_1) + e^{-\frac{k t_2}{2}} \sin(\gamma t_2) \right|. \end{aligned}$$

As a consequence,

$$\begin{aligned}
 & |G(t_1, s) - G(t_2, s)| \\
 & \leq \frac{1}{\gamma} (|h_l(s)| + 1) e^{-\frac{ks}{2}} \left| -e^{-\frac{kt_1}{2}} \sin(\gamma t_1) + e^{-\frac{kt_2}{2}} \sin(\gamma t_2) \right| \\
 & \quad + \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| e^{-\frac{kt_1}{2}} \cos(\gamma t_1) - e^{-\frac{kt_2}{2}} \cos(\gamma t_2) \right|.
 \end{aligned}$$

■ If $s \geq \xi_{m-1}$, then

$$\begin{aligned}
 & |G(t_1, s) - G(t_2, s)| \\
 & = \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| e^{-\frac{kt_1}{2}} \sin(\gamma(s - t_1)) - e^{-\frac{kt_2}{2}} \sin(\gamma(s - t_2)) \right| \\
 & \leq \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| e^{-\frac{kt_1}{2}} \cos(\gamma t_1) - e^{-\frac{kt_2}{2}} \cos(\gamma t_2) \right| \\
 & \quad + \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| -e^{-\frac{kt_1}{2}} \sin(\gamma t_1) + e^{-\frac{kt_2}{2}} \sin(\gamma t_2) \right|.
 \end{aligned}$$

Therefore, as in previous case, we can affirm that for a given $\varepsilon > 0$ there exists some $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then, for $s \in [0, t_2]$, it holds that

$$|G(t_1, s) - G(t_2, s)| \leq \varepsilon e^{-\frac{ks}{2}}.$$

This implies that the first term of the last part of inequality (7.3.4) tends to zero with independence of the function $u \in B$.

3) Finally, for $0 \leq t_2 \leq s \leq t_1$:

■ If $\xi_{l-1} \leq s < \xi_l$ for some $2 \leq l \leq m - 1$, then

$$\begin{aligned}
 & |G(t_1, s) - G(t_2, s)| \\
 & \leq \frac{1}{\gamma} e^{-\frac{ks}{2}} |h_l(s)| \left| -e^{-\frac{kt_1}{2}} \sin(\gamma t_1) + e^{-\frac{kt_2}{2}} \sin(\gamma t_2) \right| \\
 & \quad + \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| e^{-\frac{kt_1}{2}} \sin(\gamma(s - t_1)) \right|.
 \end{aligned}$$

■ If $s \geq \xi_{m-1}$, then

$$|G(t_1, s) - G(t_2, s)| = \frac{1}{\gamma} e^{-\frac{ks}{2}} \left| e^{-\frac{kt_1}{2}} \sin(\gamma(s - t_1)) \right|.$$

Thus, when $s \in [t_2, t_1]$, it holds that

$$|G(t_1, s) - G(t_2, s)| \leq \tilde{C} e^{-\frac{ks}{2}},$$

for some positive constant \tilde{C} . This implies that

$$|G(t_1, \cdot) - G(t_2, \cdot)| (\varphi_r(\cdot) + (k + M)r) \in L^1[t_1, t_2]$$

for any $t_1, t_2 \in [0, \infty)$. Then it is clear that

$$\int_{t_2}^{t_1} |G(t_1, s) - G(t_2, s)| (\varphi_r(s) + (k + M)r) ds \xrightarrow{t_1 \rightarrow t_2} 0$$

with independence of the function $u \in B$.

Thus we conclude that given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies that $|Tu(t_1) - Tu(t_2)| < \varepsilon$ for all $u \in B$.

In a completely analogous way, finding suitable upper bounds for

$$\left| \frac{\partial G}{\partial t}(t_1, s) - \frac{\partial G}{\partial t}(t_2, s) \right|,$$

it is possible to prove that given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|t_1 - t_2| < \delta$, then $|(Tu)'(t_1) - (Tu)'(t_2)| < \varepsilon$ for all $u \in B$.

Therefore, $T(B)$ is equicontinuous.

(iii) $T(B)$ is equiconvergent at ∞ :

Given $u \in B$, it holds that

$$\begin{aligned} & \left| Tu(t) - \lim_{t \rightarrow \infty} Tu(t) \right| \\ & \leq C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) ds + \left(2 + \frac{2M}{k} \right) r \right) \\ & \quad + \lim_{t \rightarrow \infty} C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) ds + \left(2 + \frac{2M}{k} \right) r \right) \\ & \leq C_1 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) ds + \left(2 + \frac{2M}{k} \right) r \right) \end{aligned}$$

and, since $e^{-\frac{ks}{2}} \varphi_r(s) \in L^1[0, \infty)$, it occurs that for every $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that if $t > N$, then

$$\left| Tu(t) - \lim_{t \rightarrow \infty} Tu(t) \right| < \varepsilon.$$

Analogously, we have that

$$\begin{aligned}
 & \left| (Tu)'(t) - \lim_{t \rightarrow \infty} (Tu)'(t) \right| \\
 & \leq C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) \\
 & \quad + \lim_{t \rightarrow \infty} C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right) \\
 & \leq C_2 e^{-\frac{kt}{2}} \left(\int_0^\infty e^{-\frac{ks}{2}} \varphi_r(s) \, ds + \left(2 + \frac{2M}{k} \right) r \right),
 \end{aligned}$$

from where we deduce that for every $\varepsilon > 0$ there exists some $\tilde{N} \in \mathbb{R}$ such that it $t > \tilde{N}$, then

$$\left| (Tu)'(t) - \lim_{t \rightarrow \infty} (Tu)'(t) \right| < \varepsilon.$$

Thus, $T(B)$ is equiconvergent at ∞ .

Therefore, from Theorem 1 in Page 181, we conclude that $T(B)$ is relatively compact in X . \square

Now we will see our existence results.

Theorem 7.3.2. *Let $f : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be such that there exists $t_0 \in [0, \infty)$ for which $f(t_0, 0, 0) \neq 0$. Moreover, suppose that, for C_1 and C_2 given in Remark 7.2.1, either*

- (H_1) holds and, moreover, there exists some $R > 0$ such that

$$\begin{aligned}
 & \max\{C_1, C_2\} \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds \right\} \\
 & + \max\{C_1, C_2\} \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}} \right) \right\} \left(1 + \frac{M}{k} \right) R < R,
 \end{aligned} \tag{7.3.5}$$

or

- (H_2) holds and, moreover, there is $R > 0$ such that

$$\begin{aligned}
 & \max\{C_1, C_2\} \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \phi_R(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \phi_R(s) \, ds \right\} \\
 & + \max\{C_1, C_2\} \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}} \right) \right\} \left(1 + \frac{M}{k} \right) R < R.
 \end{aligned}$$

Then problem (7.1.1) has at least a nontrivial solution u such that $\|u\| \leq R$.

Proof. We will prove the first case, being the second one analogous.

Consider

$$D = \{u \in X : \|u\| < R\}.$$

If $u \in D$ then,

$$|Tu(t)| \leq \int_0^\infty |G(t, s)| (\varphi_R(s) + (k + M) R) \, ds, \quad \forall t \in [0, \infty),$$

and, since $G(t, s) = 0$ for $s \geq \max\{t, \xi_{m-1}\}$,

$$|Tu(t)| \leq \int_0^{\max\{t, \xi_{m-1}\}} |G(t, s)| (\varphi_R(s) + (k + M) R) \, ds, \quad \forall t \in [0, \infty).$$

If $t > \xi_{m-1}$, previous expression leads to

$$\begin{aligned} |Tu(t)| &\leq \int_0^t |G(t, s)| (\varphi_R(s) + (k + M) R) \, ds \\ &\leq C_1 e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} (\varphi_R(s) + (k + M) R) \, ds \\ &\leq C_1 \left(e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds + 2 e^{-\frac{kt}{2}} \left(1 - e^{-\frac{kt}{2}}\right) \left(1 + \frac{M}{k}\right) R \right) \\ &\leq C_1 \left(e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds + \frac{1}{2} \left(1 + \frac{M}{k}\right) R \right). \end{aligned}$$

On the other hand, if $t \leq \xi_{m-1}$, we obtain that

$$\begin{aligned} |Tu(t)| &\leq \int_0^{\xi_{m-1}} |G(t, s)| (\varphi_R(s) + (k + M) R) \, ds \\ &\leq C_1 e^{-\frac{kt}{2}} \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} (\varphi_R(s) + (k + M) R) \, ds \\ &\leq C_1 \left(e^{-\frac{kt}{2}} \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds + 2 e^{-\frac{kt}{2}} \left(1 - e^{-\frac{k\xi_{m-1}}{2}}\right) \left(1 + \frac{M}{k}\right) R \right) \\ &\leq C_1 \left(\int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds + 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}}\right) \left(1 + \frac{M}{k}\right) R \right). \end{aligned}$$

Therefore,

$$\begin{aligned} |Tu(t)| &\leq C_1 \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds \right\} \\ &\quad + C_1 \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}}\right) \right\} \left(1 + \frac{M}{k}\right) R, \quad \forall t \in [0, \infty). \end{aligned}$$

Analogously, it can be seen that

$$|(Tu)'(t)| \leq C_2 \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds \right\} \\ + C_2 \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}} \right) \right\} \left(1 + \frac{M}{k} \right) R, \quad \forall t \in [0, \infty).$$

Thus, by (7.3.5),

$$\|Tu\| \\ \leq \max\{C_1, C_2\} \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{kt}{2}} \int_0^t e^{-\frac{ks}{2}} \varphi_R(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{ks}{2}} \varphi_R(s) \, ds \right\} \\ + \max\{C_1, C_2\} \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k\xi_{m-1}}{2}} \right) \right\} \left(1 + \frac{M}{k} \right) R < R,$$

that is, $Tu \in D$.

Therefore, $T(D) \subset D$ and, from Schauder's fixed point Theorem (Theorem 1.2.3), operator T has at least one fixed point in D , which is a solution of problem (7.1.1).

Moreover, since there exists at least some $t_0 \in [0, \infty)$ for which $f(t_0, 0, 0) \neq 0$, this solution can not be the trivial one. \square

Now, we will give another existence result based on the lower and upper solutions technique. The proof will follow the line of [145]. Before formulating the theorem, we will give a previous lemma that we will use in the proof.

Lemma 7.3.3 ([145, Lemma 2]). *Let $u \in C^1(I)$, $v, w \in W^{2,1}(I)$ and define*

$$p(t, u(t)) = \begin{cases} v(t), & u(t) < v(t), \\ u(t), & v(t) \leq u(t) \leq w(t), \\ w(t), & u(t) > w(t). \end{cases}$$

Then, the two following properties hold:

1. $\frac{d}{dt} p(t, u(t))$ exists for a. e. $t \in I$.
2. If $u, u_m \in C^1(I)$ and u_m converges to u in $C^1(I)$, then

$$\lim_{m \rightarrow \infty} \frac{d}{dt} p(t, u_m(t)) = \frac{d}{dt} p(t, u(t)), \quad \text{a. e. } t \in I.$$

Theorem 7.3.4. Let $\alpha, \beta \in X$ be lower and upper solutions of problem (7.1.1), respectively, with

$$\alpha(t) \leq \beta(t), \quad \forall t \in [0, \infty),$$

and denote

$$\tilde{R} = \max\{\|\alpha\|_\infty, \|\beta\|_\infty, \|\alpha'\|_\infty, \|\beta'\|_\infty\}. \quad (7.3.6)$$

Assume that, for C_1 and C_2 given in Remark 7.2.1, either

- (H_1) holds and, moreover, there exists some $R > 0$ such that

$$\begin{aligned} & \max\{C_1, C_2\} \max\left\{\sup_{t > \xi_{m-1}} e^{-\frac{k}{2}t} \int_0^t e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) \, ds\right\} \\ & + \max\{C_1, C_2\} \max\left\{\frac{1}{2}, 2\left(1 - e^{-\frac{k}{2}\xi_{m-1}}\right)\right\} \left(1 + \frac{M}{k}\right) R < R. \end{aligned}$$

or

- (H_2) holds and, moreover, there exists some $R > 0$ such that

$$\begin{aligned} & \max\{C_1, C_2\} \max\left\{\sup_{t > \xi_{m-1}} e^{-\frac{k}{2}t} \int_0^t e^{-\frac{k}{2}s} \phi_{\max\{R, \tilde{R}\}}(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{k}{2}s} \phi_{\max\{R, \tilde{R}\}}(s) \, ds\right\} \\ & + \max\{C_1, C_2\} \max\left\{\frac{1}{2}, 2\left(1 - e^{-\frac{k}{2}\xi_{m-1}}\right)\right\} \left(1 + \frac{M}{k}\right) R < R. \end{aligned}$$

Then, problem (7.1.1) has a solution $u \in X$ such that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \forall t \in [0, \infty).$$

Proof. We will prove the first case, being the second one analogous.

Let $\varepsilon > 0$ be such that

$$\begin{aligned} & \max\{C_1, C_2\} \max\left\{\sup_{t > \xi_{m-1}} e^{-\frac{k}{2}t} \int_0^t e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) \, ds\right\} \\ & + \max\{C_1, C_2\} \max\left\{\frac{1}{2}, 2\left(1 - e^{-\frac{k}{2}\xi_{m-1}}\right)\right\} \left(\left(1 + \frac{M}{k}\right) R + \frac{\varepsilon}{k} (R + \tilde{R})\right) < R. \end{aligned}$$

Consider the modified problem

$$\begin{cases} u''(t) + k u'(t) + M u(t) = f\left(t, \delta(t, u(t)), \frac{d}{dt} \delta(t, u(t))\right) + k u'(t) \\ \quad + M u(t) + \varepsilon (u(t) - \delta(t, u(t))), \quad t \in [0, \infty), \\ u(0) = 0, \quad u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases} \quad (7.3.7)$$

where the function $\delta: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\delta(t, u(t)) = \begin{cases} \beta(t), & u(t) > \beta(t), \\ u(t), & \alpha(t) \leq u(t) \leq \beta(t), \\ \alpha(t), & u(t) < \alpha(t). \end{cases}$$

Define now operator $T^*: X \rightarrow X$ by

$$\begin{aligned} T^*u(t) &= \int_0^\infty G(t, s) f\left(s, \delta(s, u(s)), \frac{d}{ds} \delta(s, u(s))\right) ds \\ &\quad + \int_0^\infty G(t, s) (k u'(s) + M u(s) + \varepsilon(u(s) - \delta(s, u(s)))) ds. \end{aligned}$$

From Lemma 7.3.3, we know that $\frac{d}{ds} \delta(s, u(s))$ exists for a.e. $t \in [0, \infty)$. Thus, Lemma 7.3.3 together with the Carathéodory condition on the nonlinearity, implies that the first integral in previous expression is well-defined.

Following the same steps than in Lemma 7.3.1, it is easy to prove that if (H_1) holds, then T^* is well-defined in X and it is a compact operator.

Moreover, by (7.3.6), it is clear that

$$|\delta(t, u(t))| \leq \tilde{R} \quad \text{and} \quad \left| \frac{d}{dt} \delta(t, u(t)) \right| \leq \max\{\|u\|, \tilde{R}\} \quad \text{for all } t \in [0, \infty).$$

Thus, if we consider

$$D = \{u \in X : \|u\| < R\}$$

and $u \in D$ then, following analogous steps to the proof of Theorem 7.3.2, it can be deduced that

$$\begin{aligned} &\|T^*u\| \\ &\leq \max\{C_1, C_2\} \max \left\{ \sup_{t > \xi_{m-1}} e^{-\frac{k}{2}t} \int_0^t e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) ds, \int_0^{\xi_{m-1}} e^{-\frac{k}{2}s} \varphi_{\max\{R, \tilde{R}\}}(s) ds \right\} \\ &\quad + \max\{C_1, C_2\} \max \left\{ \frac{1}{2}, 2 \left(1 - e^{-\frac{k}{2}\xi_{m-1}} \right) \right\} \left(\left(1 + \frac{M}{k} \right) R + \frac{\varepsilon}{k} (R + \tilde{R}) \right) < R, \end{aligned}$$

that is, $T^*u \in D$.

Therefore, $T(D) \subset D$ and, from Theorem 1.2.3, T^* has at least one fixed point in D , which is a solution of problem (7.3.7).

Finally, we will prove that this solution u of the modified problem (7.3.7) satisfies that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \forall t \in [0, \infty),$$

which implies that it is also a solution of problem (7.3.1).

Define $v(t) = u(t) - \beta(t)$ and consider $t_0 \in [0, \infty) \cup \{\infty\}$ such that

$$v(t_0) := \sup\{v(t) : t \in [0, \infty)\}.$$

Suppose that $v(t_0) > 0$. Then, since

$$v(0) = -\beta(0) \leq 0,$$

necessarily $t_0 \neq 0$. Thus, there exists $\bar{t} \in [0, \infty)$ such that

$$v(t) > 0 \quad \text{for } t \in (\bar{t}, t_0) \quad \text{and} \quad v'(\bar{t}) \geq 0.$$

Now, using the facts that u is a solution of (7.3.7) and β is an upper solution of problem (7.1.1), we obtain for $t \in (\bar{t}, t_0)$ that

$$\begin{aligned} u''(t) &= f\left(t, \delta(t, u(t)), \frac{d}{dt} \delta(t, u(t))\right) + \varepsilon(u(t) - \delta(t, u(t))) \\ &= f(t, \beta(t), \beta'(t)) + \varepsilon(u(t) - \beta(t)) \geq \beta''(t) + \varepsilon(u(t) - \beta(t)). \end{aligned}$$

Thus, we deduce that

$$v''(t) = u''(t) - \beta''(t) \geq \varepsilon(u(t) - \beta(t)) = \varepsilon v(t) > 0, \quad t \in (\bar{t}, t_0),$$

which implies that v' is strictly increasing on (\bar{t}, t_0) . In particular, since $v'(\bar{t}) \geq 0$, it occurs that $v' > 0$ on (\bar{t}, t_0) .

Now, the fact that $v' > 0$ on (\bar{t}, t_0) implies that $t_0 = \infty$ and so

$$\lim_{t \rightarrow \infty} v'(t) > 0.$$

On the other hand, since $v \in X$,

$$v(\infty) = \sup_{t \in [0, \infty)} v(t) = C \in \mathbb{R}$$

and, using L'Hôpital's Rule,

$$\begin{aligned} C &= \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \frac{e^t v(t)}{e^t} = \lim_{t \rightarrow \infty} \frac{e^t v(t) + e^t v'(t)}{e^t} = \lim_{t \rightarrow \infty} v(t) + v'(t) \\ &= C + \lim_{t \rightarrow \infty} v'(t), \end{aligned}$$

and we deduce that

$$\lim_{t \rightarrow \infty} v'(t) = 0,$$

which is a contradiction.

Therefore

$$\sup\{v(t) : t \in [0, \infty)\} \leq 0,$$

that is,

$$u(t) \leq \beta(t), \quad t \in [0, \infty).$$

Analogously, it can be seen that

$$u(t) \geq \alpha(t), \quad t \in [0, \infty).$$

This way, we conclude that u is a solution of problem (7.1.1). □

7.4. An Example

Let us consider the following boundary value problem:

$$\begin{cases} u''(t) = \frac{1}{1000} (2 + \sin t) e^{-|u(t)|} \frac{|1 - u(t)|}{(u(t))^2 + 1} (u'(t) - 1), & t \in [0, \infty), \\ u(0) = 0, \quad u'(+\infty) = 0.11 u'(0) + 0.89 u'(0.11). \end{cases} \quad (7.4.1)$$

This problem is a particular case of (7.1.1) with

$$f(t, x, y) = \frac{1}{1000} (2 + \sin t) e^{-|x|} \frac{|1 - x|}{x^2 + 1} (y - 1),$$

$m = 3, \alpha_1 = 0.11, \alpha_2 = 0.89, \xi_1 = 0$ and $\xi_2 = 0.11$.

We have that for $|x|, |y| < r$, it holds that

$$|f(t, x, y)| \leq \frac{1}{1000} (2 + \sin t) (r + 1)^2,$$

so we could take $\phi_r(t) = \frac{1}{1000} (2 + \sin t) (r + 1)^2$ and hypothesis (H_2) holds. We note that, since $\phi_r \notin L^1[0, \infty)$, results in [85] can not be applied to solve this problem.

Now, we will look for a pair of lower and upper solutions of problem (7.1.1) and suitable values for k and M for which the hypotheses in Theorem 7.3.4 hold.

As lower and upper solutions we will take

$$\alpha(t) = \frac{3}{400} \left(-(t + 1) e^{-t} + \frac{t^2 - t}{t^2 + 1} \right) \quad \text{and} \quad \beta(t) = 1, \quad \forall t \in [0, \infty).$$

It can be checked that $\|\alpha\|_\infty \approx 0.0087$, $\|\alpha'\|_\infty \approx 0.0065$ and $\|\beta\|_\infty = 1$, $\|\beta'\|_\infty = 0$. Therefore, we obtain that \tilde{R} given in (7.3.6) is

$$\tilde{R} = 1.$$

Moreover, for $M = 0.35$ and $k = 0.86$, we obtain the following approximations for C_1 and C_2 :

$$C_1 \approx 1.2305, \quad C_2 \approx 1.3395.$$

Therefore,

$$\max\{C_1, C_2\} \max\left\{\frac{1}{2}, 2\left(1 - e^{-\frac{k\xi_{m-1}}{2}}\right)\right\}\left(1 + \frac{M}{k}\right) \approx 0.9423.$$

On the other hand,

$$\int_0^{\xi_{m-1}} e^{-\frac{k s}{2}} \phi_{\max\{R, \tilde{R}\}}(s) \, ds \approx 0.00022 \left(\max\{R, \tilde{R}\} + 1\right)^2$$

and

$$\sup_{t > \xi_{m-1}} e^{-\frac{k t}{2}} \int_0^t e^{-\frac{k s}{2}} \phi_{\max\{R, \tilde{R}\}}(s) \, ds \approx 0.00174 \left(\max\{R, \tilde{R}\} + 1\right)^2.$$

This way, we may approximate

$$\begin{aligned} & \max\{C_1, C_2\} \max\left\{\sup_{t > \xi_{m-1}} e^{-\frac{k t}{2}} \int_0^t e^{-\frac{k s}{2}} \phi_{\max\{R, \tilde{R}\}}(s) \, ds, \int_0^{\xi_{m-1}} e^{-\frac{k s}{2}} \phi_{\max\{R, \tilde{R}\}}(s) \, ds\right\} \\ & + \max\{C_1, C_2\} \max\left\{\frac{1}{2}, 2\left(1 - e^{-\frac{k\xi_{m-1}}{2}}\right)\right\}\left(1 + \frac{M}{k}\right) R \\ & \approx 0.00233 \left(\max\{R, 1\} + 1\right)^2 + 0.9423 R, \end{aligned}$$

and it can be seen that for $R \in (R_0, R_1)$, with $R_0 \approx 0.1615$ and $R_1 \approx 22.7199$, it holds that

$$0.00233 \left(\max\{R, 1\} + 1\right)^2 + 0.9423 R < R.$$

Thus, we have proved the existence of a solution u of problem (7.4.1) such that

$$\frac{3}{400} \left(-(t+1)e^{-t} + \frac{t^2 - t}{t^2 + 1} \right) \leq u(t) \leq 1, \quad \forall t \in [0, \infty).$$

Since $\alpha(t) > 0$ for $t \in (t_1, \infty)$, with $t_1 \approx 2.01$, this solution is nontrivial.



Existence of Solutions of Integral Equations with Asymptotic Conditions

8.1. Introduction

In this chapter we will study the existence of fixed points of integral operators of the form

$$Tu(t) = p(t) + \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds,$$

that is, defined on unbounded domains.

As it has been mentioned before, the main problem when dealing with this kind of problems arise as a consequence of the lack of compactness of the domain, which causes difficulties for proving the compactness of the operator. In particular, when the domain is unbounded, it is not possible to apply Ascoli-Arzelà's Theorem (Theorem 1.2.2) to prove the compactness of the operator, as we have made in Chapters 4, 5 and 6. The most common way of solving this problem involves the compactness criterion given in Theorem 1 in Page 181, which has been used in Chapter 7.

In this chapter, we will deal with the problem of compactness of the integral operator using a different strategy: we will define a suitable Banach space, which will be proved to be isometrically isomorphic to the compact space

$$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) := \left\{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} : f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}), \exists \lim_{t \rightarrow \pm\infty} f^{(j)}(t) \in \mathbb{R}, j = 0, \dots, n \right\}.$$

This isomorphism will allow us to apply Ascoli-Arzelà's Theorem to our Banach space instead of using Theorem 1.

Moreover, the Banach space that we will define will include some condition which will ensure a certain asymptotic behavior of the solutions of the problem.

Once we have constructed the aforementioned Banach space and proved the compactness of the integral operator T in such Banach space, we will show the existence

of fixed points of T via two different approaches. First of them, which will be given in Section 8.4, will be based on the fixed point index theory in abstract cones. The second one, developed in Section 8.5, will make use of spectral theory. As we will show later with two examples, our two methods are not comparable but complementary, making it possible to deal with different kinds of differential and integral problems defined on unbounded domains, either with more restrictive conditions on the linear part (the kernel, k) or on the nonlinear one (the nonlinearity, f).

The chapter is divided in the following way: in Section 8.2 we present a physical problem which motivates the importance of the asymptotic behavior of solutions of a differential equation. In Section 8.3 we first summarize some classical definitions of asymptotic behavior and then define a suitable Banach space (in which we will look for solutions of our integral problem) and study its properties. Section 8.4 includes results of existence of fixed points of integral equations by means of the theory of fixed point index in abstract cones. In particular, in Subsection 8.4.1 we will reconsider the physical problem presented in Section 8.2 and we will solve it by using the results given in Section 8.4. It can be seen that these examples could not be solved with the method developed in Section 8.5. Finally, Section 8.5 contains results of existence of fixed points of integral equations via spectral theory and shows an example which can be solved with this theory but not with the one given in Section 8.4.

All the results regarding the method developed in Section 8.4 are collected in [34], meanwhile all the results in Section 8.5 can be found in [33].

8.2. Motivation

In many contexts it is interesting to anticipate the asymptotic behavior of the solution of a differential problem. For instance, consider the classical projectile equation that describes the motion of an object that is launched vertically from the surface of a planet towards deep space (see [77]). This situation is modelled with the following equation:

$$\begin{cases} u''(t) = -\frac{gR^2}{(u(t) + R)^2}, & t \in [0, \infty), \\ u(0) = 0, \quad u'(0) = v_0, \end{cases} \quad (8.2.1)$$

where u denotes the distance from the surface of the planet, R is the radius of the planet, g is the surface gravity constant and v_0 the initial velocity.

Clearly, if v_0 is not big enough, the projectile will reach a maximum height, at which u' will be zero, and then fall. Hence, in order to compute the minimum initial velocity necessary for the projectile to escape the planet's gravity (which is called

escape velocity), it is enough to consider that, in such a case,

$$\lim_{t \rightarrow \infty} u(t) = \infty \quad \text{and} \quad v_{\infty} := \lim_{t \rightarrow \infty} u'(t) = 0.$$

Then, multiplying both sides of the equation in (8.2.1) by u' and integrating between 0 and t , we obtain

$$\frac{1}{2} ((u'(t))^2 - v_0^2) = g R^2 \left(\frac{1}{R + u(t)} - \frac{1}{R} \right).$$

Thus, taking the limit when $t \rightarrow \infty$, we get $-v_0^2/2 = -gR$, that is, the escape velocity is

$$v_s = \sqrt{2gR}.$$

Observe that, with $v_0 = v_s$, we have that

$$u'(t) = \sqrt{\frac{2gR^2}{u(t) + R}}.$$

Using the same argument, for any initial velocity higher than v_s , when the projectile is far enough from the planet, it should drift away at constant velocity given by

$$v_{\infty} = \sqrt{v_0^2 - 2gR}.$$

Moreover, the solution of (8.2.1) has a very interesting asymptotic behavior. For $v_0 > v_s$, as it was previously said, it is asymptotically linear. This can be checked using L'Hôpital's rule:

$$\lim_{t \rightarrow \infty} \frac{u(t)}{t} = \lim_{t \rightarrow \infty} u'(t) = v_{\infty}.$$

On the other hand, in the particular case $v_0 = v_s$ we have that $v_{\infty} = 0$ and

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{u(t)}{t^{\frac{2}{3}}} &= \left[\lim_{t \rightarrow \infty} \frac{u(t)^{\frac{3}{2}}}{t} \right]^{\frac{2}{3}} = \left[\frac{3}{2} \lim_{t \rightarrow \infty} u(t)^{\frac{1}{2}} u'(t) \right]^{\frac{2}{3}} \\ &= \left[\frac{3}{2} \lim_{t \rightarrow \infty} u(t)^{\frac{1}{2}} \sqrt{\frac{2gR^2}{u(t) + R}} \right]^{\frac{2}{3}} = \left[\frac{3}{2} \sqrt{2gR^2} \right]^{\frac{2}{3}} = \left(\frac{3}{2} \right)^{\frac{2}{3}} \sqrt[3]{2gR^2}. \end{aligned}$$

In a more realistic setting, with a self propelled projectile, we could consider the following equation to model its motion:

$$\begin{cases} u''(t) = -\frac{gR^2}{(u(t) + R)^2} + h(t, u(t)) - \rho(u(t)) u'(t), & t \in [0, \infty), \\ u(0) = 0, \quad u'(0) = v_0, \end{cases} \quad (8.2.2)$$

where $h(t, y)$ is the acceleration generated by the propulsion system of the rocket (which depends on time and also height, since different phases of the launch require different propulsion systems) and ρ is the friction coefficient, which depends on height since it is related to atmospheric drag. The friction term is expected to not affect the asymptotic behavior of the solution (the atmosphere is finite, and therefore ρ has compact support), so it would be interesting to study for what kinds of h it would be reasonable to expect the same asymptotic behavior as that of the solution of (8.2.1). In any case, we would have to define first what we understand by *asymptotic behavior*.

8.3. Asymptotic Behavior

8.3.1. Classical Ways of Dealing with Asymptotic Behavior

Asymptotic behavior, always associated to perturbation theory in Physics, has been studied for a long time in an abstract mathematical way. For instance, if we go to the book of G. H. Hardy *Orders of Infinity* ([73]), we find the following notions:

“Let us suppose that f and φ are two functions of the continuous variable x , defined for all values of x greater than a given value x_0 . Let us suppose further that f and φ are positive, continuous, and steadily increasing functions which tend to infinity with x ; and let us consider the ratio f/φ . We must distinguish four cases:

- *If $f/\varphi \rightarrow \infty$ with x , we shall say that the rate of increase, or simply the increase, of f is greater than that of φ , and shall write*

$$f \succ \varphi.$$

- *If $f/\varphi \rightarrow 0$, we shall say that the increase of f is less than that of φ , and write*

$$f \prec \varphi.$$

- *If f/φ remains, for all values of x however large, between two fixed positive numbers δ, Δ , so that $0 < \delta < f/\varphi < \Delta$, we shall say that the increase of f is equal to that of φ , and write*

$$f \asymp \varphi.$$

It may happen, in this case, that f/φ actually tends to a definite limit. If this is so, we shall write

$$f \equiv \varphi.$$

Finally, if this limit is unity, we shall write

$$f \sim \varphi.$$

- If a positive constant d can be found such that $f > d\varphi$ for all sufficiently large values of x , we shall write

$$f \succ \varphi;$$

and if a positive constant Δ can be found such that $f < \Delta\varphi$ for all sufficiently large values of x , we shall write

$$f \preccurlyeq \varphi."$$

Hence, it is clear that there are several ways to approach this issue. The case of $f \preccurlyeq \varphi$ (also written as $f = O(\varphi)$ in the notation of Landau) is the one used in the study of computational complexity (see [135]).

On the other hand, we find this kind of asymptotic behavior in *fading memory spaces* (see [95]), but also in *weighted spaces* (see [134]), where the comportment can also be that associated to $f \prec \varphi$, noted as $f = o(\varphi)$ as well (see [135]).

The aforementioned notions of asymptotic behavior are connected through the exponential map to their corresponding ones using the difference instead of the quotient. To be explicit, consider the exponential map

$$\begin{aligned} \exp: \mathcal{C}(\mathbb{R}, \mathbb{R}) &\longrightarrow \mathcal{C}(\mathbb{R}, \mathbb{R}^+) \\ f &\longmapsto e^f, \end{aligned}$$

where $\mathbb{R}^+ = (0, \infty)$. Thus, for every $f, \varphi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$,

- $\lim_{x \rightarrow \infty} (f - \varphi) = \infty$ if and only if $e^f \succ e^\varphi$.
- $\lim_{x \rightarrow \infty} (f - \varphi) = -\infty$ if and only if $e^f \prec e^\varphi$.
- $|f - \varphi|$ is bounded if and only if $e^f \asymp e^\varphi$.
- $\lim_{x \rightarrow \infty} (f - \varphi) = L \in \mathbb{R}$ if and only if $e^f \asymp e^\varphi$.
- $\lim_{x \rightarrow \infty} (f - \varphi) = 0$ if and only if $e^f \sim e^\varphi$.
- A constant $d \in \mathbb{R}$ can be found such that $f - \varphi > d$ for all sufficiently large values of x if and only if $e^f \succ e^\varphi$.

- A constant $\Delta \in \mathbb{R}$ can be found such that $f - \varphi < \Delta$ for all sufficiently large values of x if and only if $e^f \preccurlyeq e^\varphi$.

Needless to say, the all of the aforementioned definitions can be applied to non necessarily positive functions with due precautions.

In this work we will center our discussion in the case $f \preccurlyeq \varphi$. In order to do so, we will need a conveniently defined Banach space.

8.3.2. The Space of Continuously n -Differentiable φ -Extensions to Infinity

Consider the space $\overline{\mathbb{R}} := [-\infty, \infty]$ with the compact topology, that is, the topology generated by the basis

$$\{B(a, r) : a \in \mathbb{R}, r \in \mathbb{R}^+\} \cup \{[-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty] : a \in \mathbb{R}\},$$

where, as usual, $B(a, r) = (a - r, a + r)$.

With this topology, $\overline{\mathbb{R}}$ is homeomorphic to any compact interval of \mathbb{R} with the relative topology inherited from the usual topology of \mathbb{R} .

It is easy to check that $\mathcal{C}(\overline{\mathbb{R}}, \mathbb{R})$ is a Banach space with the usual supremum norm. In a similar way, for $n \in \mathbb{N}$, we define

$$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) := \left\{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} : f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}), \exists \lim_{t \rightarrow \pm\infty} f^{(j)}(t) \in \mathbb{R}, j = 0, \dots, n \right\}.$$

It holds that $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$, $n \in \mathbb{N}$, is a Banach space with the norm

$$\|f\|_{(n)} := \sup \left\{ \|f^{(k)}\|_{\infty} : k = 0, \dots, n \right\}.$$

Take now $\varphi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$ and define the *space of continuously n -differentiable φ -extensions to infinity*

$$\tilde{\mathcal{C}}_{\varphi}^n \equiv \tilde{\mathcal{C}}_{\varphi}^n(\mathbb{R}, \mathbb{R}) = \left\{ f \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}) : \exists \tilde{f} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}), f = \varphi \cdot \tilde{f}|_{\mathbb{R}} \right\}.$$

We define the induced norm

$$\|f\|_{\varphi} := \|\tilde{f}\|_{(n)}, \quad f \in \tilde{\mathcal{C}}_{\varphi}.$$

Remark 8.3.1. $\|\cdot\|_{\varphi}$ is well-defined, since the extension \tilde{f} is unique for every f . Indeed, assume there are \tilde{f}_1, \tilde{f}_2 such that $\tilde{f}_1 \varphi = \tilde{f}_2 \varphi = f$ in \mathbb{R} . Since \mathbb{R} is dense in $\overline{\mathbb{R}}$ and \tilde{f}_1 and \tilde{f}_2 are continuous, $\tilde{f}_1 = \tilde{f}_2$.

On the other hand, for every $\tilde{f} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ there exists a unique $f \in \tilde{\mathcal{C}}_{\varphi}^n$ such that $\tilde{f}|_{\mathbb{R}} \varphi = f$ (just define $f := \tilde{f} \varphi$ in \mathbb{R}).

The previous remark shows that there is an isometric isomorphism between the Banach spaces $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ and $\tilde{\mathcal{C}}_\varphi^n$, namely

$$\begin{aligned}\Phi : \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) &\longrightarrow \tilde{\mathcal{C}}_\varphi^n \\ \tilde{f} &\longmapsto \Phi(\tilde{f}) = \tilde{f}|_{\mathbb{R}} \varphi,\end{aligned}$$

whose inverse isomorphism is

$$\begin{aligned}\Phi^{-1} : \tilde{\mathcal{C}}_\varphi^n &\longrightarrow \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) \\ f &\longmapsto \Phi^{-1}(f) = f/\varphi.\end{aligned}$$

Furthermore, Ascoli-Arzelà's Theorem (Theorem 1.2.2) applies to $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ since $\overline{\mathbb{R}}$ is a Hausdorff compact topological space and \mathbb{R} is a complete metric space. Using Φ we can apply the theorem to $\tilde{\mathcal{C}}_\varphi^n$.

If we write this result in terms of $\tilde{\mathcal{C}}_\varphi^n$ using the isomorphism Φ we get the following theorem.

Theorem 8.3.2. *$F \subset \tilde{\mathcal{C}}_\varphi^n$ has compact closure if and only if the two following conditions are satisfied:*

- *For each $t \in \mathbb{R}$, the set $\{\tilde{f}(t), f \in F\}$ has compact closure or, which is the same (since $\tilde{f}(t) \in \mathbb{R}$), $\{\tilde{f}(t), f \in F\}$ is bounded, that is, for each $t \in \mathbb{R}$ there exists some constant $M > 0$ such that*

$$\left| \frac{\partial^j \tilde{f}}{\partial t^j}(t) \right| = \left| \frac{\partial^j (f/\varphi)}{\partial t^j}(t) \right| \leq M < \infty,$$

for all $j = 0, \dots, n$ and $f \in F$.

- *F is equicontinuous, that is, for all $\varepsilon \in \mathbb{R}^+$ there exists some $\delta \in \mathbb{R}^+$ such that*

$$\left| \frac{\partial^j \tilde{f}}{\partial t^j}(r) - \frac{\partial^j \tilde{f}}{\partial t^j}(s) \right| = \left| \frac{\partial^j (f/\varphi)}{\partial t^j}(r) - \frac{\partial^j (f/\varphi)}{\partial t^j}(s) \right| < \varepsilon,$$

for all $j = 0, \dots, n$, $f \in F$ and $r, s \in \mathbb{R}$ such that $|r - s| < \delta$.

Proof. Let

$$\tilde{F}^{(j)} := \left\{ (f/\varphi)^{(j)} : f \in F \right\} \subset \mathcal{C}(\overline{\mathbb{R}}, \mathbb{R}), \quad j = 0, \dots, n.$$

Since

$$\|f\|_{(n)} := \sup \left\{ \|f^{(k)}\|_\infty : k = 0, \dots, n \right\},$$

F has compact closure in $\tilde{\mathcal{C}}_\varphi^n$ if and only if $\tilde{F}^{(j)}$ have compact closure in $\mathcal{C}(\overline{\mathbb{R}}, \mathbb{R})$ for $j = 0, \dots, n$. By Ascoli-Arzelà's Theorem, this happens if and only if:

- For each $t \in \mathbb{R}$, the set $\{f(t) : f \in F^{(j)}\}$ has compact closure for every $j = 0, \dots, n$.
- $\tilde{F}^{(j)}$ is equicontinuous for $j = 0, \dots, n$.

□

Remark 8.3.3. Observe that, if $f \in \mathcal{C}(\overline{\mathbb{R}}, \mathbb{R})$ and $f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R})$, then

$$\lim_{t \rightarrow \pm\infty} f^{(k)}(t) = 0 \quad \text{for every } k = 1, \dots, n$$

since f is asymptotically constant. Hence, $f \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$.

Remark 8.3.4. Although $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ and $\tilde{\mathcal{C}}_{\varphi}^n$ are isometrically isomorphic as Banach spaces, $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ is a Banach algebra but $\tilde{\mathcal{C}}_{\varphi}^n$ is not. In fact, we have that $\tilde{\mathcal{C}}_{\varphi}^n$ is a $\mathcal{C}^n(\mathbb{R}, \mathbb{R})$ -module satisfying that

$$\begin{aligned} \|fg\|_{\varphi} &\leq \max_{j=0, \dots, n} \sum_{k=0}^j \binom{j}{k} \|(f/\varphi)^{(k)}\|_{\infty} \|g^{(j-k)}\|_{\infty} \leq 2^n \|f/\varphi\|_{(n)} \|g\|_{(n)} \\ &= 2^n \|f\|_{\varphi} \|g\|_{(n)} \end{aligned}$$

for every $f \in \tilde{\mathcal{C}}_{\varphi}^n$, $g \in \mathcal{C}^n(\mathbb{R}, \mathbb{R})$.

Similarly, we can work on intervals of the form $[a, \infty)$ (or $(-\infty, a]$) instead of \mathbb{R} . In such a case we obtain the Banach space $\tilde{\mathcal{C}}_{\varphi}^n([a, \infty))$ (or $\tilde{\mathcal{C}}_{\varphi}^n((-\infty, a])$).

It is easy to construct an inclusion of $\tilde{\mathcal{C}}_{\varphi}^n([a, \infty))$ into $\tilde{\mathcal{C}}_{\varphi}^n$ using cutoff functions, so $\tilde{\mathcal{C}}_{\varphi}^n([a, \infty))$ is a Banach subspace of $\tilde{\mathcal{C}}_{\varphi}^n$.

Moreover, it is important to point out that the function φ given to define $\tilde{\mathcal{C}}_{\varphi}^n$ is not unique. In fact, we can always choose another one with better properties than the given φ .

Theorem 8.3.5. The following assertions hold:

1. For every $\varphi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$ there exists $\psi \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^+)$ such that $\tilde{\mathcal{C}}_{\varphi}^n = \tilde{\mathcal{C}}_{\psi}^n$.
2. Let $\varphi_1, \varphi_2 \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$. If $\tilde{\mathcal{C}}_{\varphi_1}^k = \tilde{\mathcal{C}}_{\varphi_2}^k$ for some $k \in \{0, \dots, n\}$, then $\tilde{\mathcal{C}}_{\varphi_1}^j = \tilde{\mathcal{C}}_{\varphi_2}^j$ for every $j \in \{0, \dots, n\}$.

Proof. 1. For every $k \in \mathbb{Z}$, let

$$\varepsilon_k := \frac{\min \varphi|_{[k, k+1]}}{|k| + 1}.$$

The Weierstrass Approximation Theorem guarantees the existence of

$$\varphi_k \in \mathcal{C}^\infty([k, k+1], \mathbb{R}^+)$$

such that

$$\|\varphi|_{[k, k+1]} - \varphi_k\|_\infty < \min\{\varepsilon_k, \varepsilon_{k-1}\}. \quad (8.3.1)$$

Let $k \in \mathbb{Z}$ be fixed. We know that φ is continuous at k , so there is $\delta_k \in (0, \frac{1}{2})$ such that

$$|\varphi(t) - \varphi(k)| < \min\{\varepsilon_k, \varepsilon_{k-1}\}, \quad \text{for every } t \in [k - \delta_k, k + \delta_k]. \quad (8.3.2)$$

Now, define

$$\rho(t) := \begin{cases} 1, & t = 0, \\ e^{-\frac{t^2}{1-t^2}} \left(1 - e^{-\frac{1}{t^2}}\right), & t \in (0, 1), \\ 0, & t = 1. \end{cases}$$

It is easy to check that $\rho \in \mathcal{C}^\infty([0, 1], [0, 1])$, $\rho^{(j)}(0) = \rho^{(j)}(1) = 0$, $j \in \mathbb{N}$. Now consider the functions

$$\psi_k(t) := \begin{cases} \varphi_k(k + \delta_k) + \int_{k+\delta_k}^t \varphi'_k(s) \rho\left(\frac{k+\delta_k-s}{\gamma_{k,1}}\right) ds, & t \in [a_k, k + \delta_k], \\ \varphi_k(t), & t \in (k + \delta_k, k + 1 - \delta_{k+1}), \\ \varphi_k(k + 1 - \delta_{k+1}) + \int_{k+1-\delta_{k+1}}^t \varphi'_{k-1}(s) \rho\left(\frac{s-(k+1-\delta_{k+1})}{\gamma_{k,2}}\right) ds, & t \in [k + 1 - \delta_{k+1}, b_k], \end{cases}$$

for every $k \in \mathbb{N}$, where

$$a_k := k + \delta_k - \gamma_{k,1},$$

$$b_k := k + 1 - \delta_{k+1} + \gamma_{k,2},$$

and

$$\gamma_{k,1} \in (0, \delta_k), \quad \gamma_{k,2} \in (0, \delta_{k+1}).$$

We have that $\psi_k \in \mathcal{C}^\infty([a_k, b_k])$ and $\psi_k^{(j)}(a_k) = \psi_k^{(j)}(b_k) = 0$ for every $j \in \mathbb{N}$.

Also, for $\gamma_{k,1}$ and $\gamma_{k,2}$ sufficiently small, we have that

$$|\varphi_k(t) - \psi_k(t)| < \min\{\varepsilon_k, \varepsilon_{k-1}\}, \quad \text{for } t \in [a_k, b_k]. \quad (8.3.3)$$

Hence, define

$$\psi(t) := \begin{cases} \psi_k(t), & t \in [a_k, b_k], \quad k \in \mathbb{Z}, \\ \psi_k(b_k) + [\psi_{k+1}(a_{k+1}) - \psi_k(b_k)] \rho\left(\frac{a_{k+1}-t}{a_{k+1}-b_k}\right), & t \in [b_k, a_{k+1}), \quad k \in \mathbb{Z}. \end{cases}$$

It holds that $\psi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ and in the sets $[b_k, a_{k+1}]$, $k \in \mathbb{Z}$, we have that

$$\begin{aligned} |\psi(t) - \varphi(t)| &= \left| \psi_k(b_k) + [\psi_{k+1}(a_{k+1}) - \psi_k(b_k)] \rho \left(\frac{a_{k+1}-t}{a_{k+1}-b_k} \right) - \varphi(t) \right| \\ &= \left| \psi_k(b_k) - \varphi_k(b_k) + [\psi_{k+1}(a_{k+1}) - \psi_k(b_k)] \rho \left(\frac{a_{k+1}-t}{a_{k+1}-b_k} \right) + \varphi_k(b_k) - \varphi(b_k) + \varphi(b_k) - \varphi(t) \right| \\ &\leq |\psi_k(b_k) - \varphi_k(b_k)| + |\psi_{k+1}(a_{k+1}) - \psi_k(b_k)| + |\varphi_k(b_k) - \varphi(b_k)| + |\varphi(b_k) - \varphi(t)|. \end{aligned}$$

Now, using (8.3.1) and (8.3.3),

$$\begin{aligned} |\psi(t) - \varphi(t)| &< 2\varepsilon_k + |\psi_{k+1}(a_{k+1}) - \psi_k(b_k)| + |\varphi(b_k) - \varphi(t)| \\ &\leq 2\varepsilon_k + |\psi_{k+1}(a_{k+1}) - \psi_k(b_k)| + |\varphi(b_k) - \varphi(k+1)| + |\varphi(k+1) - \varphi(t)|, \end{aligned}$$

and, from (8.3.2),

$$\begin{aligned} |\psi(t) - \varphi(t)| &< 4\varepsilon_k + |\psi_{k+1}(a_{k+1}) - \psi_k(b_k)| \\ &\leq 4\varepsilon_k + |\psi_{k+1}(a_{k+1}) - \varphi_{k+1}(a_{k+1})| + |\varphi_{k+1}(a_{k+1}) - \varphi_k(b_k)| + |\varphi_k(b_k) - \psi_k(b_k)|. \end{aligned}$$

Using again (8.3.3), we get that

$$\begin{aligned} |\psi(t) - \varphi(t)| &< 6\varepsilon_k + |\varphi_{k+1}(a_{k+1}) - \varphi_k(b_k)| \\ &\leq 6\varepsilon_k + |\varphi_{k+1}(a_{k+1}) - \varphi(a_{k+1})| + |\varphi(a_{k+1}) - \varphi(k+1)| + |\varphi(k+1) - \varphi(b_k)| \\ &\quad + |\varphi(b_k) - \varphi_k(b_k)|, \end{aligned}$$

and, from (8.3.1) and (8.3.2), we conclude that

$$|\psi(t) - \varphi(t)| < 10\varepsilon_k = 10 \frac{\min \varphi|_{[k, k+1]}}{|k|+1}, \quad \text{for every } t \in [k, k+1], \quad k \in \mathbb{Z}.$$

Now, for every $t \in [k, k+1]$, $|k| > 10$,

$$|\psi(t)| > |\varphi(t)| - 10 \frac{\min \varphi|_{[k, k+1]}}{|k|+1}.$$

Thus,

$$\begin{aligned} \frac{|\varphi(t) - \psi(t)|}{|\psi(t)|} &< \frac{10 \min \varphi|_{[k, k+1]}}{(|k|+1)|\psi(t)|} < \frac{10 \min \varphi|_{[k, k+1]}}{(|k|+1) \left(|\varphi(t)| - \frac{10 \min \varphi|_{[k, k+1]}}{|k|+1} \right)} \\ &= \frac{10}{(|k|+1) \frac{|\varphi(t)|}{\min \varphi|_{[k, k+1]}} - 10} \leq \frac{10}{|k| - 9}. \end{aligned}$$

This fact allows us to prove that

$$\lim_{t \rightarrow \pm\infty} \left| \frac{\varphi(t)}{\psi(t)} - 1 \right| = \lim_{t \rightarrow \pm\infty} \frac{|\varphi(t) - \psi(t)|}{|\psi(t)|} \leq \lim_{|k| \rightarrow \infty} \frac{10}{|k| - 9} = 0.$$

Hence,

$$\lim_{t \rightarrow \pm\infty} \frac{\varphi(t)}{\psi(t)} = \lim_{t \rightarrow \pm\infty} \frac{\psi(t)}{\varphi(t)} = 1.$$

Therefore, if $f \in \tilde{\mathcal{C}}_\varphi^n$,

$$\lim_{t \rightarrow \pm\infty} \frac{f(t)}{\psi(t)} = \lim_{t \rightarrow \pm\infty} \frac{f(t)}{\varphi(t)} \frac{\varphi(t)}{\psi(t)} = \lim_{t \rightarrow \pm\infty} \frac{f(t)}{\varphi(t)} \lim_{t \rightarrow \pm\infty} \frac{\varphi(t)}{\psi(t)} = \tilde{f}(\pm\infty).$$

Thus, $f \in \tilde{\mathcal{C}}_\psi^n$. The other inclusion is analogous, so $\tilde{\mathcal{C}}_\varphi^n = \tilde{\mathcal{C}}_\psi^n$.

2. By definition, $\varphi_2 \in \tilde{\mathcal{C}}_{\varphi_2}^k = \tilde{\mathcal{C}}_{\varphi_1}^k$, so there exists $\tilde{f}_{\varphi_2} \in \mathcal{C}^k(\overline{\mathbb{R}}, \mathbb{R})$ such that $\varphi_2 = \varphi_1 \tilde{f}_{\varphi_2}|_{\mathbb{R}}$. By Remark 8.3.3, we have that $\tilde{f}_{\varphi_2} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$.

Hence, for $j \in \{0, \dots, n\}$ and $f \in \tilde{\mathcal{C}}_{\varphi_2}^j$, there exists $\tilde{f}_2 \in \mathcal{C}^j(\overline{\mathbb{R}}, \mathbb{R})$ such that $f = \varphi_2 \tilde{f}_2|_{\mathbb{R}} = \varphi_1 (\tilde{f}_{\varphi_2} \tilde{f}_2)|_{\mathbb{R}}$. Therefore, we show that $f \in \tilde{\mathcal{C}}_{\varphi_1}^j$ and thus $\tilde{\mathcal{C}}_{\varphi_2}^j \subset \tilde{\mathcal{C}}_{\varphi_1}^j$. Analogously, $\tilde{\mathcal{C}}_{\varphi_1}^j \subset \tilde{\mathcal{C}}_{\varphi_2}^j$ and so $\tilde{\mathcal{C}}_{\varphi_2}^j = \tilde{\mathcal{C}}_{\varphi_1}^j$. □

Remark 8.3.6. Theorem 8.3.5 allows us to consider spaces of the form $\tilde{\mathcal{C}}_\varphi^n$ even when $\varphi \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ is not differentiable. In order to do so, we just pick a function $\psi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ to represent the space $\tilde{\mathcal{C}}_\varphi^n = \tilde{\mathcal{C}}_\psi^n$ and consider $\tilde{\mathcal{C}}_\psi^n$. Furthermore, Theorem 8.3.5 implies that $\tilde{\mathcal{C}}_\psi^n$ does not depend of the choice of ψ .

8.3.3. Comparison with Other Spaces

Take $\varphi \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$. We can define two types of weighted spaces:

$$\mathcal{C}_\varphi^*(\mathbb{R}) := \left\{ f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \lim_{x \rightarrow \pm\infty} \frac{f(x)}{\varphi(x)} = 0 \right\}$$

and

$$CB_\varphi(\mathbb{R}) := \left\{ f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \frac{f}{\varphi} \text{ is bounded} \right\}.$$

Both of them are Banach spaces with the norm $\|f\|_\varphi = \left\| \frac{f}{\varphi} \right\|_\infty$.

Furthermore, define

$$\varphi_1(x) := \begin{cases} 0, & x > 1, \\ \min \{ \varphi(x), \varphi(0)(1-x) \}, & x \in [0, 1], \\ \varphi(x), & x < 0, \end{cases}$$

and

$$\varphi_2(x) := \begin{cases} 0, & x < 0, \\ \min \{ \varphi(x), \varphi(1)x \}, & x \in [0, 1], \\ \varphi(x), & x > 1. \end{cases}$$

φ_1 and φ_2 are continuous and we may define the linear map

$$\begin{aligned} \Xi: \mathcal{C}_\varphi^*(\mathbb{R}) \oplus \mathbb{R}^2 &\longrightarrow \tilde{\mathcal{C}}_\varphi(\mathbb{R}) \\ (f, a, b) &\longmapsto f + a\varphi_1 + b\varphi_2. \end{aligned}$$

If we define on $\mathcal{C}_\varphi^*(\mathbb{R}) \oplus \mathbb{R}^2$ the norm

$$\|(f, a, b)\| = \|f\|_\varphi + |a| + |b|,$$

we have that $\|\Xi(f, a, b)\|_\varphi \leq \|(f, a, b)\|$, so Ξ is continuous.

Moreover, Ξ is invertible. Just consider

$$\Xi^{-1}(f) = \left(f - \left(\lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right) \varphi_1 - \left(\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right) \varphi_2, \lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)}, \lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right),$$

and, furthermore,

$$\begin{aligned} &\|\Xi^{-1}(f)\| \\ &= \left\| f - \left(\lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right) \varphi_1 - \left(\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right) \varphi_2 \right\|_\varphi + \left| \lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right| + \left| \lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right| \\ &\leq \|f\|_\varphi + \left| \lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right| \|\varphi_1\|_\varphi + \left| \lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right| \|\varphi_2\|_\varphi + \left| \lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right| + \left| \lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right| \\ &\leq \|f\|_\varphi + 2 \left| \lim_{x \rightarrow -\infty} \frac{f(x)}{\varphi(x)} \right| + 2 \left| \lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} \right| \leq 5 \|f\|_\varphi, \end{aligned}$$

so $\mathcal{C}_\varphi^*(\mathbb{R}) \oplus \mathbb{R}^2$ and $\tilde{\mathcal{C}}_\varphi(\mathbb{R})$ are isomorphic Banach spaces (up to equivalent norms).

On the other hand, $\tilde{\mathcal{C}}_\varphi(\mathbb{R})$ is a closed subspace of $CB_\varphi(\mathbb{R})$. Unfortunately $\tilde{\mathcal{C}}_\varphi(\mathbb{R})$ has no clear complement subspace in $CB_\varphi(\mathbb{R})$ as in the case of $\mathcal{C}_\varphi^*(\mathbb{R})$ in $CB_\varphi(\mathbb{R})$.

Furthermore, in the case of $CB_\varphi(\mathbb{R})$, the domain of its functions is unbounded, but cannot be compactified in a meaningful way preserving the behavior of the functions, so we would have to use Theorem 1 instead of Theorem 8.3.2 in order to work with it.

Moreover, when looking for solutions of differential or integral problems, it is clear that if we assure that these solutions are in the space $\tilde{\mathcal{C}}_\varphi^n$, we will have a much

more precise information about their asymptotic behavior than when we find them in the space $CB_\varphi(\mathbb{R})$. This reason, together with the possibility of applying Ascoli-Arzelà's Theorem, is why we find more convenient to use the space $\tilde{\mathcal{C}}_\varphi^n$ instead of $CB_\varphi(\mathbb{R})$.

8.4. Fixed Points of Integral Equations

In this section we will develop our first method to prove the existence of fixed points of the integral operator T given by

$$Tu(t) := p(t) + \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds. \quad (8.4.1)$$

As we have mentioned before, this method will be based on the fixed point index theory on abstract cones. In particular, given $\varphi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$, we will work with cones of functions in the space $\tilde{\mathcal{C}}_\varphi^n$. This way, we will obtain solutions with a certain asymptotic behavior.

To do this, we will follow the line of [56], where the authors studied the existence of solutions of integral equations of Hammerstein-type in abstract cones. In particular, they considered a real normed space $(N, \|\cdot\|)$ and a continuous functional $\alpha: N \rightarrow \mathbb{R}$. They proved that if this functional α satisfies the three following properties:

$$(P_1) \quad \alpha(u + v) \geq \alpha(u) + \alpha(v), \text{ for all } u, v \in N;$$

$$(P_2) \quad \alpha(\lambda u) \geq \lambda \alpha(u), \text{ for all } u \in N, \lambda \geq 0;$$

$$(P_3) \quad [\alpha(u) \geq 0, \alpha(-u) \geq 0] \Rightarrow u \equiv 0;$$

then

$$K_\alpha = \{u \in N : \alpha(u) \geq 0\}$$

is a cone.

Following their arguments, we will consider the cone

$$K_\alpha = \left\{ u \in \tilde{\mathcal{C}}_\varphi^n : \alpha(u) \geq 0 \right\},$$

where $\alpha: \tilde{\mathcal{C}}_\varphi^n \rightarrow \mathbb{R}$ is a functional satisfying $(P_1) - (P_3)$.

Moreover, we will make the following assumptions:

$$(C_1) \quad \text{The kernel } k: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \text{ is such that } \frac{\partial^j k}{\partial t^j}(t, \cdot) \eta(\cdot) \in L^1(\mathbb{R}) \text{ for every } t \in \mathbb{R}, \\ j = 0, \dots, n \text{ and } k(\cdot, s) \eta(s) \in \tilde{\mathcal{C}}_\varphi^n \text{ for every } s \in \mathbb{R}.$$

Moreover, for every $\varepsilon > 0$ and $j = 0, \dots, n$, there exist $\delta > 0$ and a measurable function ω_j such that if $|t_1 - t_2| < \delta$ then

$$\left| \frac{\partial^j(k/\varphi)}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j(k/\varphi)}{\partial t^j}(t_2, s) \eta(s) \right| < \varepsilon \omega_j(s)$$

for a. e. $s \in \mathbb{R}$.

(C₂) $f: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ satisfies a sort of L^1 -Carathéodory conditions, that is:

- $f(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a. e. $t \in \mathbb{R}$.
- For each $r > 0$, there exists $\phi_r \in L^1(\mathbb{R})$ such that

$$f(t, y \varphi(t)) \leq \phi_r(t)$$

for all $y \in [-r, r]$ and a. e. $t \in \mathbb{R}$.

(C₃) For every fixed $r > 0$, $j = 0, \dots, n$ and $l = 0, \dots, j$,

$$\frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) \right| \phi_r(s) \, ds \in L^\infty(\mathbb{R})$$

and $\omega_j \phi_r \in L^1(\mathbb{R})$.

Moreover, defining

$$z^\pm(s) := \lim_{t \rightarrow \pm\infty} \frac{k(t, s) \eta(s)}{\varphi(t)}$$

and

$$M_j(s) := \sup_{t \in \mathbb{R}} \left| \frac{\partial^j(k/\varphi)}{\partial t^j}(t, s) \eta(s) \right|,$$

these functions must satisfy that $|z^\pm| \phi_r, M_j \phi_r \in L^1(\mathbb{R})$ for all $r > 0$.

(C₄) $p \in \tilde{\mathcal{C}}_\varphi^n$.

(C₅) $\alpha(k(\cdot, s) \eta(s)) \geq 0$ for a. e. $s \in \mathbb{R}$ and $\alpha(p) \geq 0$.

(C₆)

$$\alpha(Tu) \geq \int_{-\infty}^{\infty} \alpha(k(\cdot, s) \eta(s)) f(s, u(s)) \, ds + \alpha(p) \text{ for all } u \in K_\alpha.$$

(C₇) There exist two continuous functionals $\beta, \gamma: \tilde{\mathcal{C}}_\varphi^n \rightarrow \mathbb{R}$ satisfying that, for $u, v \in K_\alpha$ and $\lambda \in [0, \infty)$,

$$\beta(\lambda u) = \lambda \beta(u), \quad \beta(Tu) \leq \int_{-\infty}^{\infty} \beta(k(\cdot, s) \eta(s)) f(s, u(s)) \, ds + \beta(p)$$

and

$$\begin{aligned} \gamma(u + v) &\geq \gamma(u) + \gamma(v), \quad \gamma(\lambda u) \geq \lambda \gamma(u), \\ \gamma(Tu) &\geq \int_{-\infty}^{\infty} \gamma(k(\cdot, s) \eta(s)) f(s, u(s)) \, ds + \gamma(p). \end{aligned}$$

Moreover, for all $s \in \mathbb{R}$, $\beta(k(\cdot, s) \eta(s)), \gamma(k(\cdot, s) \eta(s)) \in L^1(\mathbb{R})$ must be positive.

(C₈) There exists $\xi \in K_\alpha \setminus \{0\}$ such that $\gamma(\xi) \geq 0$.

(C₉) For every $\rho > 0$ there exist either $b(\rho) > 0$ such that $\beta(u) \leq b(\rho)$ for every $u \in K_\alpha$ satisfying $\gamma(u) \leq \rho$ or $c(\rho) > 0$ such that $\gamma(u) \leq c(\rho)$ for every $u \in K_\alpha$ satisfying $\beta(u) \leq \rho$.

Theorem 8.4.1. Assume that hypotheses (C₁)–(C₆) hold. Then T maps $(\tilde{\mathcal{C}}_\varphi^n, \|\cdot\|_\varphi)$ to itself, is continuous and compact and maps K_α to K_α .

Proof. We will divide the proof into several steps.

Step 1: T maps $(\tilde{\mathcal{C}}_\varphi^n, \|\cdot\|_\varphi)$ to $(\tilde{\mathcal{C}}_\varphi^n, \|\cdot\|_\varphi)$:

Given $u \in \tilde{\mathcal{C}}_\varphi^n$, we will see that $Tu \in \tilde{\mathcal{C}}_\varphi^n$.

Let $j \in \{0, \dots, n\}$. By condition (C₁), we can use Leibniz's Integral Rule to get

$$\frac{\partial^j \widetilde{Tu}}{\partial t^j}(t) = \frac{\partial^j (Tu/\varphi)}{\partial t^j}(t) = \int_{-\infty}^{\infty} \frac{\partial^j (k/\varphi)}{\partial t^j}(t, s) \eta(s) f(s, u(s)) \, ds + \frac{\partial^j (p/\varphi)}{\partial t^j}(t). \quad (8.4.2)$$

On the other hand, from condition (C₁), given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\begin{aligned} &\left| \frac{\partial^j \widetilde{k}}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j \widetilde{k}}{\partial t^j}(t_2, s) \eta(s) \right| \\ &= \left| \frac{\partial^j (k/\varphi)}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j (k/\varphi)}{\partial t^j}(t_2, s) \eta(s) \right| < \varepsilon \omega_j(s), \end{aligned}$$

and since $p \in \tilde{\mathcal{C}}_\varphi^n$,

$$\left| \frac{\partial^j \tilde{p}}{\partial t^j}(t_1) - \frac{\partial^j \tilde{p}}{\partial t^j}(t_2) \right| = \left| \frac{\partial^j (p/\varphi)}{\partial t^j}(t_1) - \frac{\partial^j (p/\varphi)}{\partial t^j}(t_2) \right| < \varepsilon.$$

Therefore, from (C_2) ,

$$\begin{aligned} & \left| \frac{\partial^j \tilde{T}u}{\partial t^j}(t_1) - \frac{\partial^j \tilde{T}u}{\partial t^j}(t_2) \right| \\ & \leq \int_{-\infty}^{\infty} \left| \frac{\partial^j (k/\varphi)}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j (k/\varphi)}{\partial t^j}(t_2, s) \eta(s) \right| f(s, u(s)) \, ds \\ & \quad + \left| \frac{\partial^j (p/\varphi)}{\partial t^j}(t_1) - \frac{\partial^j (p/\varphi)}{\partial t^j}(t_2) \right| \\ & \leq \varepsilon \left(\int_{-\infty}^{\infty} \omega_j(s) f(s, u(s)) \, ds + 1 \right) \leq \varepsilon \left(\int_{-\infty}^{\infty} \omega_j(s) \phi_{\|u\|_\varphi}(s) \, ds + 1 \right), \end{aligned} \tag{8.4.3}$$

where we have used the fact that

$$f(s, u(s)) = f\left(s, \frac{u(s)}{\varphi(s)} \varphi(s)\right) \leq \phi_{\|u\|_\varphi}(s), \quad \text{a.e. } s \in \mathbb{R}.$$

Since $\omega_j \phi_{\|u\|_\varphi} \in L^1(\mathbb{R})$, the previous expression is upperly bounded by εc for some positive constant c . Hence, $\frac{\partial^j \tilde{T}u}{\partial t^j}$ is continuous in \mathbb{R} , that is, $\tilde{T}u \in \mathcal{C}^n(\mathbb{R}, \mathbb{R})$.

Now we will see that there exists (and is a real number) the limit of $\tilde{T}u$ when t goes to $\pm\infty$. First, note that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \tilde{T}u(t) &= \lim_{t \rightarrow \pm\infty} \frac{Tu(t)}{\varphi(t)} \\ &= \lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds + \lim_{t \rightarrow \pm\infty} \frac{p(t)}{\varphi(t)}. \end{aligned}$$

Since $p, k(\cdot, s) \eta(s) \in \tilde{\mathcal{C}}_\varphi^n$ for all $s \in \mathbb{R}$, there exist

$$\lim_{t \rightarrow \pm\infty} \frac{p(t)}{\varphi(t)} \in \mathbb{R} \quad \text{and} \quad \lim_{t \rightarrow \pm\infty} \frac{k(t, s) \eta(s)}{\varphi(t)} = z^\pm(s) \in \mathbb{R}.$$

On the other hand,

$$\left| \frac{k(t, s) \eta(s)}{\varphi(t)} f(s, u(s)) \right| \leq M_0(s) f(s, u(s)) \leq M_0(s) \phi_{\|u\|_\varphi}(s) \quad \text{for all } t \in \mathbb{R}$$

and, from (C_3) , $M_0 \phi_{\|u\|_\varphi} \in L^1(\mathbb{R})$. Thus, from Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds &= \int_{-\infty}^{\infty} \lim_{t \rightarrow \pm\infty} \frac{k(t, s) \eta(s)}{\varphi(t)} f(s, u(s)) \, ds \\ &= \int_{-\infty}^{\infty} z^\pm(s) f(s, u(s)) \, ds \end{aligned}$$

and, since

$$\begin{aligned} \left| \int_{-\infty}^{\infty} z^\pm(s) f(s, u(s)) \, ds \right| &\leq \int_{-\infty}^{\infty} |z^\pm(s)| |f(s, u(s))| \, ds \\ &\leq \int_{-\infty}^{\infty} |z^\pm(s)| \phi_{\|u\|_\varphi}(s) \, ds < \infty, \end{aligned}$$

we deduce that there exists $\lim_{t \rightarrow \pm\infty} \frac{Tu(t)}{\varphi(t)} \in \mathbb{R}$.

Thus, since $\widetilde{T}u = \frac{Tu}{\varphi} \in C^n(\mathbb{R}, \mathbb{R})$ and the previous limit exists, taking into account Remark 8.3.3, it is clear that Tu is bounded in $\|\cdot\|_\varphi$ and, consequently, $Tu \in \widetilde{C}_\varphi^n$.

Step 2: Continuity:

Let $\{u_n\}_{n \in \mathbb{N}} \subset \widetilde{C}_\varphi^n$ be a sequence which converges to u in \widetilde{C}_φ^n . Then, there exists some $R \in \mathbb{R}$ such that $\|u_n\|_\varphi \leq R$ for all $n \in \mathbb{N}$.

Moreover, $\lim_{n \rightarrow \infty} \|u_n - u\|_\varphi = 0$ implies that $\lim_{n \rightarrow \infty} \left\| \frac{u_n}{\varphi} - \frac{u}{\varphi} \right\|_\infty = 0$, from where we deduce that

$$\frac{u_n(s)}{\varphi(s)} \rightarrow \frac{u(s)}{\varphi(s)}, \quad \text{for a.e. } s \in \mathbb{R}.$$

Therefore, $u_n(s) \rightarrow u(s)$ for a.e. $s \in \mathbb{R}$ and we have, by virtue of (C_2) , that $f(s, u_n(s)) \rightarrow f(s, u(s))$ for a.e. $s \in \mathbb{R}$.

From (8.4.2), it is clear that, for all $t \in \mathbb{R}$ and $j \in \{0, \dots, n\}$,

$$\left| \frac{\partial^j \widetilde{T}u_n}{\partial t^j}(t) - \frac{\partial^j \widetilde{T}u}{\partial t^j}(t) \right| \leq \int_{-\infty}^{\infty} \left| \frac{\partial^j (k/\varphi)}{\partial t^j}(t, s) \eta(s) \right| |f(s, u_n(s)) - f(s, u(s))| \, ds.$$

Now, using (C_2) and (C_3) ,

$$\begin{aligned} \left| \frac{\partial^j \widetilde{T}u_n}{\partial t^j}(t) - \frac{\partial^j \widetilde{T}u}{\partial t^j}(t) \right| &\leq \int_{-\infty}^{\infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &\leq 2 \int_{-\infty}^{\infty} M_j(s) \phi_R(s) \, ds < \infty. \end{aligned}$$

Now we deduce, by application of Lebesgue's Dominated Convergence Theorem, that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \frac{\partial^j \widetilde{T}u_n}{\partial t^j} - \frac{\partial^j \widetilde{T}u}{\partial t^j} \right\|_{\infty} &\leq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &= \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds = 0. \end{aligned}$$

Therefore, we deduce that $Tu_n \rightarrow Tu$ in $\widetilde{C}_{\varphi}^n$. Hence, operator T is continuous.

Step 3: Compactness:

Let $B \subset \widetilde{C}_{\varphi}^n$ be a bounded set, that is, there exists some $R > 0$ such that $\|u\|_{\varphi} \leq R$ for all $u \in B$.

First, we will see that $T(B)$ is uniformly bounded. Using the General Leibniz's Rule (for differentiation), it is clear that

$$\frac{\partial^j \widetilde{T}u}{\partial t^j} = \frac{\partial^j (Tu/\varphi)}{\partial t^j} = \sum_{l=0}^j \binom{j}{l} \frac{\partial^l Tu}{\partial t^l} \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi}.$$

Moreover, from Leibniz's Integral Rule,

$$\frac{\partial^l Tu}{\partial t^l}(t) = \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) f(s, u(s)) \, ds + \frac{\partial^l p}{\partial t^l}(t), \quad t \in \mathbb{R}.$$

Thus,

$$\begin{aligned} &\left\| \frac{\partial^j \widetilde{T}u}{\partial t^j} \right\|_{\infty} \\ &= \left\| \sum_{l=0}^j \binom{j}{l} \frac{\partial^l Tu}{\partial t^l} \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi} \right\|_{\infty} \leq \sum_{l=0}^j \binom{j}{l} \left\| \frac{\partial^l Tu}{\partial t^l} \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi} \right\|_{\infty} \\ &= \sum_{l=0}^j \binom{j}{l} \left\| \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi(\cdot)} \left(\int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) f(s, u(s)) \, ds + \frac{\partial^l p}{\partial t^l} \right) \right\|_{\infty} \\ &\leq \sum_{l=0}^j \binom{j}{l} \left(\left\| \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi(\cdot)} \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) f(s, u(s)) \, ds \right\|_{\infty} + \left\| \frac{\partial^{j-l} 1}{\partial t^{j-l} \varphi} \frac{\partial^l p}{\partial t^l} \right\|_{\infty} \right). \end{aligned} \tag{8.4.4}$$

It is satisfied that

$$\begin{aligned}
 & \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) f(s, u(s)) \, ds \right| \\
 & \leq \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \right| \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) \right| |f(s, u(s))| \, ds \\
 & \leq \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \right| \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) \right| \phi_R(s) \, ds,
 \end{aligned} \tag{8.4.5}$$

and so, from (8.4.4) and (8.4.5) and using (C_3) and (C_4) ,

$$\left\| \frac{\partial^j \widetilde{T}u}{\partial t^j} \right\|_{\infty} \leq \sum_{l=0}^j \binom{j}{l} \left(\left\| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(\cdot) \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) \right| \phi_R(s) \, ds \right\|_{\infty} + \left\| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi} \frac{\partial^l p}{\partial t^l} \right\|_{\infty} \right) < \infty.$$

So, we have found an upper bound which does not depend on u . Therefore it is clear that the set $T(B)$ is uniformly bounded.

On the other hand, taking into account the upper bound obtained in (8.4.3), we have that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$, then, for $j = 0, \dots, n$,

$$\begin{aligned}
 \left| \frac{\partial^j \widetilde{T}u}{\partial t^j}(t_1) - \frac{\partial^j \widetilde{T}u}{\partial t^j}(t_2) \right| & \leq \varepsilon \left(\int_{-\infty}^{\infty} w_j(s) f(s, u(s)) \, ds + 1 \right) \\
 & \leq \varepsilon \left(\int_{-\infty}^{\infty} w_j(s) \phi_R(s) \, ds + 1 \right).
 \end{aligned}$$

Then, since from (C_3) $w_j \phi_R \in L^1(\mathbb{R})$, we can conclude that there exists some constant c such that

$$\left| \frac{\partial^j \widetilde{T}u}{\partial t^j}(t_1) - \frac{\partial^j \widetilde{T}u}{\partial t^j}(t_2) \right| < \varepsilon c, \quad \text{for all } u \in B.$$

This implies that $T(B)$ is equicontinuous.

In conclusion, we derive, by application of Theorem 8.3.2, that $T(B)$ is relatively compact in $\widetilde{\mathcal{C}}_{\varphi}^n$. Therefore, T is a compact operator.

Step 4: T maps K_{α} to K_{α} :

It is an immediate consequence of conditions (C_5) and (C_6) . □

Now we will give some conditions under which we can assure that the index of some subsets of K_{α} is 1 or 0. We will consider the following sets:

$$K_{\alpha}^{\beta, \rho} := \{u \in K_{\alpha} : \beta(u) < \rho\},$$

$$K_{\alpha}^{\gamma, \rho} := \{u \in K_{\alpha} : \gamma(u) < \rho\}.$$

Now, hypothesis (C_9) implies that either there exists a function $b: \mathbb{R}^+ \rightarrow \mathbb{R}$ given by

$$b(\rho) := \sup \{\beta(u) : u \in K_{\alpha}, \gamma(u) < \rho\},$$

or there exists $c: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$c(\rho) := \sup \{\gamma(u) : u \in K_{\alpha}, \beta(u) < \rho\}.$$

With these definitions, $K_{\alpha}^{\beta, \rho} \subset K_{\alpha}^{\gamma, c(\rho)}$ and $K_{\alpha}^{\gamma, \rho} \subset K_{\alpha}^{\beta, b(\rho)}$, in case that the aforementioned functions exist.

To prove that the index of some of these subsets is 1 or 0, we will use the sufficient conditions given in Lemma 1.2.7.

Lemma 8.4.2. *Assume that conditions (C_1) – (C_7) hold. Moreover let there exist $\rho > 0$ such that*

$$0 < f^{\rho} \int_{-\infty}^{\infty} \beta(k(\cdot, s) \eta(s)) \, ds + \frac{\beta(p)}{\rho} < 1, \quad (I_{\rho}^1)$$

where

$$f^{\rho} = \sup \left\{ \frac{f(t, u(t))}{\rho} : t \in \mathbb{R}, u \in K_{\alpha}, \beta(u) = \rho \right\}.$$

Then $i_{K_{\alpha}}(T, K_{\alpha}^{\beta, \rho}) = 1$.

Proof. We will prove that $Tu \neq \mu u$ for all $u \in \partial K_{\alpha}^{\beta, \rho}$ and for every $\mu \geq 1$.

Suppose, on the contrary, that there exist some $u \in \partial K_{\alpha}^{\beta, \rho}$ and $\mu \geq 1$ such that

$$\mu u(t) = \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds + p(t).$$

Then, taking β on both sides and using (C_7) , we get

$$\begin{aligned} \mu \rho &= \mu \beta(u) = \beta(Tu) \leq \int_{-\infty}^{\infty} \beta(k(\cdot, s) \eta(s)) f(s, u(s)) \, ds + \beta(p) \\ &\leq \rho \left(f^{\rho} \int_{-\infty}^{\infty} \beta(k(\cdot, s) \eta(s)) \, ds + \frac{\beta(p)}{\rho} \right) < \rho, \end{aligned}$$

which is a contradiction. Therefore, from Lemma 1.2.7, we conclude the veracity of the result. \square

Lemma 8.4.3. Assume that conditions (C_1) – (C_6) and (C_8) hold. Moreover, let there exist $\rho > 0$ such that

$$K_{\alpha}^{\gamma, \rho} \text{ is bounded and } f_{\rho} \int_{-\infty}^{\infty} \gamma(k(\cdot, s) \eta(s)) \, ds + \frac{\gamma(p)}{\rho} > 1, \quad (I_{\rho}^0)$$

where

$$f_{\rho} = \inf \left\{ \frac{f(t, u(t))}{\rho} : t \in \mathbb{R}, u \in K_{\alpha}, \gamma(u) = \rho \right\}.$$

Then $i_{K_{\alpha}}(T, K_{\alpha}^{\gamma, \rho}) = 0$.

Proof. We will prove that there exists $e \in K_{\alpha}^{\gamma, \rho} \setminus \{0\}$ such that $u \neq Tu + \lambda e$ for all $u \in \partial K_{\alpha}^{\gamma, \rho}$ and all $\lambda > 0$.

Let us take $e = \xi$ in (C_8) and suppose, on the contrary, that there exist some $u \in \partial K_{\alpha}^{\gamma, \rho}$ and $\lambda > 0$ such that

$$u(t) = \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds + p(t) + \lambda e(t).$$

Now, taking γ on both sides and using (C_7) and (C_8) ,

$$\begin{aligned} \rho &= \gamma(u) = \gamma(Tu + \lambda e) \geq \gamma(Tu) + \lambda \gamma(e) \geq \gamma(Tu) \\ &\geq \int_{-\infty}^{\infty} \gamma(k(\cdot, s) \eta(s)) f(s, u(s)) \, ds + \gamma(p) \\ &\geq \rho \left(f_{\rho} \int_{-\infty}^{\infty} \gamma(k(\cdot, s) \eta(s)) \, ds + \frac{\gamma(p)}{\rho} \right) > \rho, \end{aligned}$$

which is a contradiction. The result follows from Lemma 1.2.7. \square

From previous lemmas, it is possible to formulate the following theorem. In this case, we establish conditions to ensure the existence of one or two solutions of the integral equation (8.4.1). However, similar results can be formulated to ensure the existence of three or more solutions.

Theorem 8.4.4. Assume that conditions (C_1) – (C_9) hold. The integral equation (8.4.1) has at least one nontrivial solution in K_{α} if one of the following conditions holds:

- (S_1) There exist b given in condition (C_9) and $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_2 > b(\rho_1)$ such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold.
- (S_2) There exist c given in condition (C_9) and $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_2 > c(\rho_1)$ such that $(I_{\rho_1}^1)$ and $(I_{\rho_2}^0)$ hold.

The integral equation (8.4.1) has at least two nontrivial solutions in K_α if one of the following conditions holds:

- (S₃) There exist both b and c in condition (C₉) and $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_2 > b(\rho_1)$ and $\rho_3 > c(\rho_2)$ such that $(I_{\rho_1}^0)$, $(I_{\rho_2}^1)$ and $(I_{\rho_3}^0)$ hold.
- (S₄) There exist both b and c in condition (C₉) and $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_2 > c(\rho_1)$ and $\rho_3 > b(\rho_2)$ such that $(I_{\rho_1}^1)$, $(I_{\rho_2}^0)$ and $(I_{\rho_3}^1)$ hold.

The proof of previous theorem is immediate from Lemmas 8.4.2 and 8.4.3, together with the general properties of fixed point index given in Lemma 1.2.7.

Remark 8.4.5. We note that the previous results could also be formulated for either $\tilde{\mathcal{C}}_\varphi^n([a, \infty))$ or $\tilde{\mathcal{C}}_\varphi^n((-\infty, a])$ for any $a \in \mathbb{R}$.

8.4.1. An Example: Asymptotic Behavior of a Self Propelled Projectile

We will finally apply the theory developed in this section to solve a particular case of the problem of the self propelled projectile which has been formulated in Section 8.2, namely

$$\begin{cases} u''(t) = -\frac{gR^2}{(u(t) + R)^2} + h(t, u(t)) - \rho(u(t)) u'(t), & t \in [0, \infty), \\ u(0) = 0, \quad u'(0) = v_0. \end{cases} \quad (8.4.6)$$

As stated in Section 8.2, we will ignore the friction term (the term depending on u') because it is only related to atmospheric drag and therefore does not affect the asymptotic behavior.

Hence, we will study the problem

$$\begin{cases} u''(t) = f(t, u(t)), & t \in [0, \infty); \\ u(0) = 0, \quad u'(0) = v_0, \end{cases} \quad (8.4.7)$$

with $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined as

$$f(t, y) = -\frac{gR^2}{(y + R)^2} + h(t, y),$$

where $h : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ represents the propulsion of the projectile.

Given the domain of f and h and taking into account Remark 8.4.5, we will work on the interval $[0, \infty)$.

Rewriting (8.4.7) as an integral problem, we know that the solutions of (8.4.7) coincide with the fixed points of the following integral operator,

$$Tu(t) = p(t) + \int_0^\infty k(t, s) f(s, u(s)) \, ds, \quad (8.4.8)$$

where

$$p(t) = v_0 t$$

and

$$k(t, s) = \begin{cases} t - s, & 0 \leq s \leq t, \\ 0, & \text{otherwise,} \end{cases}$$

is the corresponding Green's function. We note that in this case the Green's function is nonnegative on $[0, \infty) \times [0, \infty)$.

We will take

$$h(s, y) = \frac{g R^2}{(y + R)^2} + y e^{-s}$$

for $s, y \in [0, \infty)$.

To ensure the constant sign of f , we extend h (and thus f) in the following way:

$$h(s, y) = \frac{g R^2}{(y + R)^2} \text{ for } y < 0.$$

We note that this extension does not have a physical meaning, as we know that the variable y will never be negative in reality, but it is considered to ensure the applicability of the results in this section in order to solve the problem.

We will consider

$$\varphi(t) = t + 1,$$

and work in the space $\tilde{\mathcal{C}}_\varphi([0, \infty))$. Our cone

$$K_\alpha = \left\{ u \in \tilde{\mathcal{C}}_\varphi([0, \infty)) : \alpha(u) \geq 0 \right\}$$

will be defined by the functional

$$\begin{aligned} \alpha: \tilde{\mathcal{C}}_\varphi([0, \infty)) &\longrightarrow \mathbb{R} \\ u &\longmapsto \alpha(u) = \int_0^\infty \frac{u(t)}{\varphi_2(t)} \, dt - \|u\|_{\varphi_3}, \end{aligned}$$

with $\varphi_2(t) = C e^t$ for some constant $C > 0$, which will be calculated later, and $\varphi_3(t) = e^t$.

The functional α is well-defined because if $u \in \tilde{\mathcal{C}}_\varphi([0, \infty))$, then it holds that $u(t) = (t+1)\tilde{u}(t)$, with $\tilde{u} \in \mathcal{C}([0, \infty], \mathbb{R})$, which implies that \tilde{u} is uniformly bounded for some constant N . Then,

$$\begin{aligned} \left| \int_0^\infty \frac{u(t)}{C e^t} dt \right| &= \left| \int_0^\infty \frac{(t+1)\tilde{u}(t)}{C e^t} dt \right| \leq \int_0^\infty \frac{t|\tilde{u}(t)|}{C e^t} dt + \int_0^\infty \frac{|\tilde{u}(t)|}{C e^t} dt \\ &\leq N \left(\int_0^\infty \frac{t}{C e^t} dt + \int_0^\infty \frac{1}{C e^t} dt \right) = \frac{2N}{C}, \end{aligned}$$

and

$$\sup_{t \in [0, \infty)} \frac{|u(t)|}{e^t} \leq \sup_{t \in [0, \infty)} \frac{|u(t)|}{t+1} = \|u\|_\varphi,$$

so $\alpha(u) \in \mathbb{R}$ for all $u \in \tilde{\mathcal{C}}_\varphi([0, \infty))$.

Moreover, it is easy to check that α satisfies properties (P_1) – (P_3) and therefore the cone K_α is well-defined:

(P_1) For all $u, v \in \tilde{\mathcal{C}}_\varphi([0, \infty))$, it holds that

$$\begin{aligned} \alpha(u+v) &= \int_0^\infty \frac{u(t)+v(t)}{\varphi_2(t)} dt - \|u+v\|_{\varphi_3} \\ &\geq \int_0^\infty \frac{u(t)}{\varphi_2(t)} dt + \int_0^\infty \frac{v(t)}{\varphi_2(t)} dt - \|u\|_{\varphi_3} - \|v\|_{\varphi_3} \\ &= \alpha(u) + \alpha(v). \end{aligned}$$

(P_2) For all $u \in \tilde{\mathcal{C}}_\varphi([0, \infty))$ and $\lambda \geq 0$,

$$\alpha(\lambda u) = \int_0^\infty \frac{\lambda u(t)}{\varphi_2(t)} dt - \|\lambda u\|_{\varphi_3} = \lambda \alpha(u).$$

(P_3) If

$$\alpha(u) = \int_0^\infty \frac{u(t)}{\varphi_2(t)} dt - \|u\|_{\varphi_3} \geq 0$$

and

$$\alpha(-u) = \int_0^\infty \frac{-u(t)}{\varphi_2(t)} dt - \|u\|_{\varphi_3} \geq 0,$$

then $-2\|u\|_{\varphi_3} \geq 0$. This implies that $\|u\|_{\varphi_3} = 0$, which is equivalent to $u \equiv 0$.

We will see now that hypotheses (C_1) – (C_9) for $n = 0$ are satisfied:

(C₁) In this case $\eta \equiv 1$ and $k(t, \cdot) \eta(\cdot) \in L^1([0, \infty))$ for every $t \in [0, \infty)$; indeed

$$\int_0^\infty |k(t, s) \eta(s)| \, ds = \int_0^t (t - s) \, ds = \frac{t^2}{2}.$$

Moreover, $k(\cdot, s) \eta(s) \in \tilde{\mathcal{C}}_\varphi([0, \infty))$ for every $s \in [0, \infty)$. This is deduced from the fact that $k(\cdot, s) \eta(s) \in \mathcal{C}([0, \infty))$ and there exist both limits

$$\lim_{t \rightarrow \infty} \frac{k(t, s) \eta(s)}{\varphi(t)} = \lim_{t \rightarrow \infty} \frac{t - s}{t + 1} = 1$$

and

$$\lim_{t \rightarrow 0} \frac{k(t, s) \eta(s)}{\varphi(t)} = 0.$$

Finally, we will see that last condition in (C₁) is satisfied for $\omega_0(s) = 1 + s$.

Fix $\varepsilon > 0$. Since $\frac{1}{\varphi}$ is a uniformly continuous function, there exists $\delta < \varepsilon$ such that for $|t_1 - t_2| < \delta$, $\left| \frac{1}{t_1 + 1} - \frac{1}{t_2 + 1} \right| < \varepsilon$. We will compute now the difference $\left| \frac{k(t_1, s)}{\varphi(t_1)} - \frac{k(t_2, s)}{\varphi(t_2)} \right|$. Fix $s \in [0, \infty)$,

- If $t_1, t_2 > s$, then

$$\begin{aligned} \left| \frac{k(t_1, s)}{\varphi(t_1)} - \frac{k(t_2, s)}{\varphi(t_2)} \right| &= \left| \frac{t_1 - s}{t_1 + 1} - \frac{t_2 - s}{t_2 + 1} \right| = \left| \frac{-1 - s}{t_1 + 1} - \frac{-1 - s}{t_2 + 1} \right| \\ &= (1 + s) \left| \frac{1}{t_1 + 1} - \frac{1}{t_2 + 1} \right| < \varepsilon \omega_0(s). \end{aligned}$$

- If $t_1 > s$ and $t_2 < s$, then

$$\left| \frac{k(t_1, s)}{\varphi(t_1)} - \frac{k(t_2, s)}{\varphi(t_2)} \right| = \left| \frac{t_1 - s}{t_1 + 1} \right| < \left| \frac{t_1 - t_2}{t_1 + 1} \right| < \frac{\varepsilon}{t_1 + 1} < \varepsilon < \varepsilon \omega_0(s).$$

- If $t_1, t_2 < s$, then

$$\left| \frac{k(t_1, s)}{\varphi(t_1)} - \frac{k(t_2, s)}{\varphi(t_2)} \right| = 0.$$

(C₂) By definition of h , we have that $f(t, y) = 0$ for $y < 0$ and $f(t, y) = y e^{-t} \geq 0$ for $y \geq 0$. Clearly, $f(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}$ and $f(t, \cdot)$ is continuous for a. e. $t \in [0, \infty)$. Finally, for each $r > 0$,

$$f(t, y \varphi(t)) = 0, \quad \text{for all } y \in [-r, 0], \, t \in [0, \infty)$$

and

$$f(t, y \varphi(t)) = y \varphi(t) e^{-t} \leq r \varphi(t) e^{-t}, \quad \text{for all } y \in [0, r], t \in [0, \infty).$$

Therefore condition (C_2) is satisfied if we take $\phi_r(t) = r \varphi(t) e^{-t}$.

(C_3) For a fixed $r > 0$, we have that

$$\begin{aligned} \frac{1}{\varphi(t)} \int_0^\infty |k(t, s) \eta(s)| \phi_r(s) \, ds &= \frac{1}{t+1} \int_0^t (t-s) r (s+1) e^{-s} \, ds \\ &= \frac{r}{t+1} (-3 + 2t + e^{-t}(3+t)), \end{aligned}$$

so

$$\frac{1}{\varphi(t)} \int_0^\infty |k(t, s) \eta(s)| \phi_r(s) \, ds \in L^\infty([0, \infty)).$$

Moreover,

$$\int_0^\infty \omega_0(s) \phi_r(s) \, ds = \int_0^\infty r (s+1)^2 e^{-s} \, ds = 5r,$$

that is, $\omega_0 \phi_r \in L^1([0, \infty))$.

Finally, from the limits calculated in (C_1) and the expression of Green's function, we have that $z^+(s) = 1$, $z^-(s) = 0$ and $M_0(s) = 1$, so it is clear that $|z^+| \phi_r, |z^-| \phi_r, M_0 \phi_r \in L^1([0, \infty))$.

(C_4) It is clear that $p(t) = v_0 t \in \tilde{\mathcal{C}}_\varphi([0, \infty))$ since $p \in \mathcal{C}([0, \infty))$ and there exist both

$$\lim_{t \rightarrow \infty} \frac{p(t)}{\varphi(t)} = v_0$$

and

$$\lim_{t \rightarrow 0} \frac{p(t)}{\varphi(t)} = 0.$$

(C_5) We have to prove that

$$\alpha(k(\cdot, s)) = \int_0^\infty \frac{k(\tau, s)}{\varphi_2(\tau)} \, d\tau - \|k(\cdot, s)\|_{\varphi_3} \geq 0 \quad \text{for a. e. } s \in [0, \infty).$$

We have that

$$\int_0^\infty \frac{k(\tau, s)}{\varphi_2(\tau)} \, d\tau = \int_s^\infty \frac{\tau - s}{C e^\tau} \, d\tau = \frac{e^{-s}}{C}.$$

On the other hand, fixed s , we have that

$$\left| \frac{k(t, s)}{e^t} \right| = 0, \quad t \leq s,$$

and

$$\left| \frac{k(t, s)}{e^t} \right| = \frac{t-s}{e^t} = e^{-s} \frac{t-s}{e^{t-s}} \leq e^{-s} e^{-1}, \quad t \geq s.$$

Therefore, it is enough to take $C \leq e$ to ensure that $\alpha(k(\cdot, s)) \geq 0$.

On the other hand,

$$\alpha(p) = \int_0^\infty \frac{p(t)}{\varphi_2(t)} dt - \|p\|_{\varphi_3} = \int_0^\infty \frac{v_0 t}{C e^t} dt - \sup_{t \in [0, \infty)} \left| \frac{v_0 t}{e^t} \right| = \frac{v_0}{C} - v_0 e^{-1}.$$

Therefore, $\alpha(p) \geq 0$ if and only if $C \leq e$.

(C₆) By definition,

$$\alpha(Tu) = \int_0^\infty \frac{Tu(t)}{\varphi_2(t)} dt - \|Tu\|_{\varphi_3}.$$

We have that

$$\begin{aligned} \int_0^\infty \frac{Tu(t)}{\varphi_2(t)} dt &= \int_0^\infty \left(\int_0^\infty \frac{k(t, s)}{\varphi_2(t)} f(s, u(s)) ds + \frac{p(t)}{\varphi_2(t)} \right) dt \\ &= \int_0^\infty \left(\int_0^\infty \frac{k(t, s)}{\varphi_2(t)} dt \right) f(s, u(s)) ds + \int_0^\infty \frac{p(t)}{\varphi_2(t)} dt, \end{aligned}$$

and

$$\begin{aligned} \|Tu\|_{\varphi_3} &= \left\| \int_0^\infty k(\cdot, s) f(s, u(s)) ds + p \right\|_{\varphi_3} \\ &\leq \left\| \int_0^\infty k(\cdot, s) f(s, u(s)) ds \right\|_{\varphi_3} + \|p\|_{\varphi_3} \\ &\leq \int_0^\infty \|k(\cdot, s)\|_{\varphi_3} f(s, u(s)) ds + \|p\|_{\varphi_3}, \end{aligned}$$

and, consequently,

$$\begin{aligned} \alpha(Tu) &\geq \int_0^\infty \left(\int_0^\infty \frac{k(t, s)}{\varphi_2(t)} dt \right) f(s, u(s)) ds \\ &\quad - \int_0^\infty \|k(\cdot, s)\|_{\varphi_3} f(s, u(s)) ds + \int_0^\infty \frac{p(t)}{\varphi_2(t)} dt - \|p\|_{\varphi_3} \\ &= \int_0^\infty \alpha(k(\cdot, s)) f(s, u(s)) ds + \alpha(p). \end{aligned}$$

(C₇) We will define $\beta, \gamma: \tilde{\mathcal{C}}_\varphi([0, \infty)) \rightarrow \mathbb{R}$ in the following way:

$$\beta(u) = \|u\|_{\varphi_3}, \text{ with } \varphi_3(t) = e^t,$$

and

$$\gamma(u) = \int_0^\infty \frac{u(t)}{e^t} dt.$$

Analogously to α , functionals β and γ are well-defined on $\tilde{\mathcal{C}}_\varphi([0, \infty))$.

Now, we will show that β and γ satisfy all the properties in condition (C₇). It is obvious that $\beta(\lambda u) = \lambda \beta(u)$ for all $\lambda \in [0, \infty)$ and $u \in K_\alpha$.

Moreover,

$$\begin{aligned} \beta(Tu) &= \|Tu\|_{\varphi_3} \leq \int_0^\infty \|k(\cdot, s)\|_{\varphi_3} f(s, u(s)) + \|p\|_{\varphi_3} \\ &= \int_0^\infty \beta(k(\cdot, s)) f(s, u(s)) + \beta(p). \end{aligned}$$

Finally, it is clear that $\beta(k(\cdot, s)) > 0$ and, since

$$\beta(k(\cdot, s)) = \sup_{t \in [s, \infty)} \frac{t-s}{e^t} = e^{-(s+1)},$$

it occurs that

$$0 < \int_0^\infty \beta(k(\cdot, s)) ds = \int_0^\infty e^{-(s+1)} ds = e^{-1},$$

that is, $\beta(k(\cdot, s)) \in L^1([0, \infty))$.

With regard to γ , it is immediate that γ is linear.

Also,

$$\begin{aligned} \gamma(Tu) &= \int_0^\infty \frac{Tu(t)}{e^t} dt = \int_0^\infty \int_0^\infty \frac{k(t, s)}{e^t} f(s, u(s)) ds dt + \int_0^\infty \frac{p(t)}{e^t} dt \\ &= \int_0^\infty \left(\int_0^\infty \frac{k(t, s)}{e^t} dt \right) f(s, u(s)) ds + \gamma(p) \\ &= \int_0^\infty \gamma(k(\cdot, s)) f(s, u(s)) ds + \gamma(p). \end{aligned}$$

Finally,

$$\gamma(k(\cdot, s)) = \int_s^\infty \frac{t-s}{e^t} dt = e^{-s} > 0, \quad s \in [0, \infty),$$

and

$$\int_0^\infty \gamma(k(\cdot, s)) \, ds = \int_0^\infty e^{-s} \, ds = 1,$$

that is, $\gamma(k(\cdot, s)) \in L^1([0, \infty))$.

(C₈) By condition (C₅) we know that $p \in K_\alpha \setminus \{0\}$. Thus, since

$$\gamma(p) = \int_0^\infty \frac{p(t)}{e^t} \, dt = \int_0^\infty \frac{v_0 t}{e^t} \, dt = v_0 > 0,$$

it is enough to take $\xi = p$.

(C₉) Every $u \in K_\alpha$ satisfies that $\beta(u) \leq \frac{1}{C} \gamma(u)$, so it is enough to define $b(\rho) = \frac{\rho}{C}$.

Note that in this case it is not possible to define a function c such that $\gamma(u) \leq c(\rho)$ for every $u \in K_\alpha$ satisfying $\beta(u) \leq \rho$.

Now, we will see that there exist some values of ρ for which (I_ρ^0) and (I_ρ^1) are satisfied.

Let's take $u \in K_\alpha$ such that $\beta(u) = \sup_{t \in [0, \infty)} \left| \frac{u(t)}{e^t} \right| = \rho$. This implies that

$$\begin{aligned} f^\rho &= \sup \left\{ \frac{u(t) e^{-t}}{\rho} : t \in [0, \infty), u \in K_\alpha, \beta(u) = \rho, u(t) \geq 0 \right\} \\ &= \frac{1}{\rho} \sup \left\{ \frac{u(t)}{e^t} : t \in [0, \infty), u \in K_\alpha, \beta(u) = \rho, u(t) \geq 0 \right\} = 1. \end{aligned}$$

Consequently,

$$f^\rho \int_0^\infty \beta(k(\cdot, s)) \, ds + \frac{\beta(p)}{\rho} = e^{-1} + \frac{v_0 e^{-1}}{\rho},$$

and (I_ρ^1) is satisfied if and only if ρ is such that $e^{-1} + \frac{v_0 e^{-1}}{\rho} < 1$, that is, if and only if

$$\rho > \frac{e^{-1}}{1 - e^{-1}} v_0 \approx 0.58197 \dots v_0.$$

On the other hand, $f_\rho \geq 0$ and so

$$f_\rho \int_0^\infty \gamma(k(\cdot, s)) \, ds + \frac{\gamma(p)}{\rho} \geq \frac{\gamma(p)}{\rho} = \frac{v_0}{\rho}.$$

Therefore, (I_ρ^0) is satisfied for all $\rho < v_0$.

Finally, we will see that there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_2 > b(\rho_1)$ such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold.

We have proved that condition (C_5) is satisfied for all $C \leq e$. Now, if we choose $C \geq e - 1$ and ρ_1, ρ_2 satisfying that

$$\rho_1 < v_0 \quad \text{and} \quad \rho_2 > v_0 \frac{1}{e-1},$$

then it is clear that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold and

$$b(\rho_1) < \frac{v_0}{C} \leq \frac{v_0}{e-1} < \rho_2.$$

Therefore, we conclude that problem (8.4.7) has at least a nontrivial solution in K_α .

We point out that this solution is in the space $\tilde{\mathcal{C}}_\varphi([0, \infty))$, which implies that its asymptotic behavior is the same than $\varphi(t) = t + 1$. This way, we are able to ensure that if the propulsion of the projectile is given by the previously defined function h , then its trajectory will be asymptotically linear (that is, the same than in the case without propulsion).

8.5. Existence of Solutions via Spectral Theory

In this section we complement the findings obtained in Section 8.4 by approaching the problem in a different way. This method will be based on the definition of some auxiliary linear operators. Then, if some limits involving the nonlinearity f and the spectral radius of the auxiliary operators satisfy some suitable properties, we will be able to find fixed points of the original integral operator.

This approach has been used successfully previously, as we can see in the works of Infante et al. ([79]), Webb and Lan ([151]) or even in the case of linearly bounded nonlinear operators as it is shown in [30].

The main advantage of this method is that, while in the previous section we had fairly restrictive conditions on the nonlinearity f , here we relax in a significant way those restrictions.

There is of course a price to pay for the advantage regarding the nonlinearity, and is that the conditions on the kernel k occurring in (8.4.1) are more restrictive. In particular, one can check that the results in the present section could not be applied, for instance, to the problem studied in Subsection 8.4.1 as its kernel does not satisfied condition (\tilde{C}_2) .

At the same time, we will show in Subsection 8.5.1 an example which is solved with the method developed in this section but does not satisfy the hypotheses required in Section 8.4.

This will prove that, as it has been mentioned in the introduction of the chapter, our two methods are not comparable but complementary.

On the other hand, we note that, for the sake of simplicity, we will not include in this section the function p occurring in (8.4.1), that is, we will work with the following integral operator

$$Tu(t) = \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds. \quad (8.5.1)$$

However, we point out that the term p could be included with minor adaptations, following the hypotheses for p given in Section 8.4.

Finally, as in Section 8.4, we will consider the abstract cone

$$K_\alpha = \left\{ u \in \tilde{\mathcal{C}}_\varphi^n : \alpha(u) \geq 0 \right\},$$

where $\alpha: \tilde{\mathcal{C}}_\varphi^n \rightarrow \mathbb{R}$ is a functional satisfying (P_1) – (P_3) .

Remark 8.5.1. *If the cone K is defined by a continuous functional α (as it will occur with the cones considered in this chapter), then v an element of the cone will belong to its interior if and only if $\alpha(v) > 0$.*

In order to state our eigenvalue comparison results, we consider the following linear operator on $\tilde{\mathcal{C}}_\varphi^n$:

$$L_1 u(t) := \int_{-\infty}^{\infty} |k(t, s) \eta(s)| u(s) \, ds, \quad t \in \mathbb{R}.$$

Moreover, we will denote by P the cone of nonnegative functions in $\tilde{\mathcal{C}}_\varphi^n$, that is

$$P := \left\{ u \in \tilde{\mathcal{C}}_\varphi^n : u \geq 0 \text{ on } \mathbb{R} \right\}.$$

In this section we will assume the following hypotheses:

(\tilde{C}_1) The kernel $k: \mathbb{R}^2 \rightarrow \mathbb{R}$, is such that $k(\cdot, s) \eta(s) \in \tilde{\mathcal{C}}_\varphi^n$ for every $s \in \mathbb{R}$.
Moreover:

- If $n = 0$, then for every $\varepsilon > 0$, there exist $\delta > 0$ and a measurable function ω_0 such that if $|t_1 - t_2| < \delta$ then

(i)

$$\left| \frac{k(t_1, s) \eta(s)}{\varphi(t_1)} - \frac{k(t_2, s) \eta(s)}{\varphi(t_2)} \right| < \varepsilon \omega_0(s)$$

and

(ii)

$$\left| \frac{(k(t_1, s) \eta(s))^+}{\varphi(t_1)} - \frac{(k(t_2, s) \eta(s))^+}{\varphi(t_2)} \right| < \varepsilon \omega_0(s),$$

for a. e. $s \in \mathbb{R}$.

Here, as usual,

$$(k(t, s) \eta(s))^+ = \max \{k(t, s) \eta(s), 0\}.$$

Moreover, we note that (i) implies that

$$\left| \frac{|k(t_1, s) \eta(s)|}{\varphi(t_1)} - \frac{|k(t_2, s) \eta(s)|}{\varphi(t_2)} \right| < \varepsilon \omega_0(s),$$

for a. e. $s \in \mathbb{R}$.

- If $n > 0$, $k(t, s) \eta(s) \geq 0$ and for every $\varepsilon > 0$ and $j = 0, \dots, n$, there exist $\delta > 0$ and a measurable function ω_j such that if $|t_1 - t_2| < \delta$ then

$$\left| \frac{\partial^j(k/\varphi)}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j(k/\varphi)}{\partial t^j}(t_2, s) \eta(s) \right| < \varepsilon \omega_j(s)$$

for a. e. $s \in \mathbb{R}$.

(\tilde{C}_2) It holds that $\omega_j \varphi, \frac{\partial^j k}{\partial t^j}(t, \cdot) \eta(\cdot) \varphi(\cdot) \in L^1(\mathbb{R})$ for every $t \in \mathbb{R}$, $j = 0, \dots, n$; and

$$\frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi} \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) \right| \varphi(s) \, ds \in L^\infty(\mathbb{R}),$$

for all $j = 0, \dots, n$ and $l = 0, \dots, j$.

Moreover, defining

$$z_{(\pm)}(s) := \lim_{t \rightarrow \pm\infty} \frac{|k(t, s) \eta(s)|}{\varphi(t)}$$

and

$$M_j(s) := \sup_{t \in \mathbb{R}} \left| \frac{\partial^j(k/\varphi)}{\partial t^j}(t, s) \eta(s) \right|,$$

it is satisfied that $z_{(\pm)} \varphi, M_j \varphi \in L^1(\mathbb{R})$, for $j = 0, \dots, n$.

(\tilde{C}_3) $f: \mathbb{R} \times \mathbb{R} \rightarrow [0, +\infty)$ satisfies a sort of L^∞ -Carathéodory conditions, that is:

- $f(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a. e. $t \in \mathbb{R}$.
- For each $r > 0$, there exists $\phi_r \in L^\infty(\mathbb{R})$ such that

$$\frac{f(t, x \varphi(t))}{\varphi(t)} \leq \phi_r(t),$$

for all $x \in [-r, r]$ and a. e. $t \in \mathbb{R}$.

(\tilde{C}_4) $\alpha(|k(\cdot, s)\eta(s)|) \geq 0$ for a. e. $s \in \mathbb{R}$.

(\tilde{C}_5) $\alpha(|k(\cdot, s)\eta(s)|) \varphi(s) \in L^1(\mathbb{R})$ and

$$\alpha(L_1 u) \geq \int_{-\infty}^{\infty} \alpha(|k(\cdot, s)\eta(s)|) u(s) \, ds \text{ for all } u \in P.$$

(\tilde{C}_6) There exists $A \subset \mathbb{R}$ such that A is a finite union of compact intervals and $k(t, s)\eta(s) \geq 0$, $k(\cdot, s)\eta(s) \not\equiv 0$ for every $t \in A$ and a. e. $s \in \mathbb{R}$. Moreover, it holds that

$$\frac{1}{\widetilde{M}(A)} = \frac{1}{\widetilde{M}} := \inf_{t \in A} \int_A k(t, s)\eta(s) \, ds > 0.$$

We will also define the following auxiliary linear operator on $\tilde{\mathcal{C}}_\varphi^n$:

$$L_2 u(t) := \int_A (k(t, s)\eta(s))^+ u(s) \, ds, \quad t \in \mathbb{R}.$$

With regard to operator L_2 , we will consider the following assumptions:

(\tilde{C}_7) $\alpha((k(\cdot, s)\eta(s))^+) \geq 0$ for a. e. $s \in \mathbb{R}$.

(\tilde{C}_8) $\alpha((k(\cdot, s)\eta(s))^+) \varphi(s) \in L^1(A)$ and

$$\alpha(L_2 u) \geq \int_A \alpha((k(\cdot, s)\eta(s))^+) u(s) \, ds \text{ for all } u \in P.$$

Finally, to ensure that operator T maps the cone K_α into itself, we need to ask for the following conditions:

(\tilde{C}_9) $\alpha(k(\cdot, s)\eta(s)) \geq 0$ for a. e. $s \in \mathbb{R}$.

(\tilde{C}_{10}) $\alpha(k(\cdot, s)\eta(s)) \varphi(s) \in L^1(\mathbb{R})$ for a. e. $s \in \mathbb{R}$ and

$$\alpha(Tu) \geq \int_{-\infty}^{\infty} \alpha(k(\cdot, s)\eta(s)) f(s, u(s)) \, ds \text{ for all } u \in K_\alpha.$$

Theorem 8.5.2. *If (\tilde{C}_1), (\tilde{C}_2), (\tilde{C}_4) and (\tilde{C}_5) hold, then operator L_1 is continuous, compact and maps P into $P \cap K_\alpha$.*

Proof. We will distinguish two different cases:

CASE I: $n = 0$:

We will divide the proof into several steps.

Step 1: L_1 maps $(\tilde{\mathcal{C}}_\varphi, \|\cdot\|_\varphi)$ to $(\tilde{\mathcal{C}}_\varphi, \|\cdot\|_\varphi)$:

Given $u \in \widetilde{\mathcal{C}}_\varphi$, we will see that $L_1 u \in \widetilde{\mathcal{C}}_\varphi$.

From $(\widetilde{\mathcal{C}}_1)$, (i), given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\begin{aligned}
 \left| \widetilde{L_1 u}(t_1) - \widetilde{L_1 u}(t_2) \right| &\leq \int_{-\infty}^{\infty} \left| \frac{|k(t_1, s) \eta(s)|}{\varphi(t_1)} - \frac{|k(t_2, s) \eta(s)|}{\varphi(t_2)} \right| |u(s)| \, ds \\
 &\leq \varepsilon \int_{-\infty}^{\infty} \omega_0(s) |u(s)| \, ds \\
 &= \varepsilon \int_{-\infty}^{\infty} \omega_0(s) \frac{|u(s)|}{\varphi(s)} \varphi(s) \, ds \\
 &\leq \varepsilon \|u\|_\varphi \int_{-\infty}^{\infty} \omega_0(s) \varphi(s) \, ds
 \end{aligned} \tag{8.5.2}$$

and since, by $(\widetilde{\mathcal{C}}_2)$, $\omega_0 \varphi \in L^1(\mathbb{R})$, the previous expression is bounded from above by $\varepsilon \|u\|_\varphi c$ for some positive constant c . Hence, $\widetilde{L_1 u}$ is continuous in \mathbb{R} .

Now we will prove that there exists $\lim_{t \rightarrow \pm\infty} \widetilde{L_1 u}(t) \in \mathbb{R}$. Indeed,

$$\lim_{t \rightarrow \pm\infty} \widetilde{L_1 u}(t) = \lim_{t \rightarrow \pm\infty} \frac{L_1 u(t)}{\varphi(t)} = \lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_{-\infty}^{\infty} |k(t, s) \eta(s)| u(s) \, ds.$$

Since $k(\cdot, s) \eta(s) \in \widetilde{\mathcal{C}}_\varphi$, then, for all $s \in \mathbb{R}$, there exists

$$\lim_{t \rightarrow \pm\infty} \frac{|k(t, s) \eta(s)|}{\varphi(t)} =: z_{(\pm)}(s) \in \mathbb{R}.$$

On the other hand, for all $t \in \mathbb{R}$ and a. e. $s \in \mathbb{R}$,

$$\left| \frac{|k(t, s) \eta(s)|}{\varphi(t)} u(s) \right| \leq M_0(s) |u(s)| = M_0(s) \frac{|u(s)|}{\varphi(s)} \varphi(s) \leq \|u\|_\varphi M_0(s) \varphi(s)$$

and, from $(\widetilde{\mathcal{C}}_2)$, $M_0 \varphi \in L^1(\mathbb{R})$. Thus, from Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
 \lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_{-\infty}^{\infty} |k(t, s) \eta(s)| u(s) \, ds &= \int_{-\infty}^{\infty} \lim_{t \rightarrow \pm\infty} \frac{|k(t, s) \eta(s)|}{\varphi(t)} u(s) \, ds \\
 &= \int_{-\infty}^{\infty} z_{(\pm)}(s) u(s) \, ds,
 \end{aligned}$$

and, since,

$$\begin{aligned}
 \left| \int_{-\infty}^{\infty} z_{(\pm)}(s) u(s) \, ds \right| &\leq \int_{-\infty}^{\infty} z_{(\pm)}(s) |u(s)| \, ds \\
 &\leq \|u\|_\varphi \int_{-\infty}^{\infty} z_{(\pm)}(s) \varphi(s) \, ds \in \mathbb{R},
 \end{aligned}$$

we deduce that $z_{(\pm)} u \in L^1(\mathbb{R})$. Therefore there exists $\lim_{t \rightarrow \pm\infty} \frac{L_1 u(t)}{\varphi(t)} \in \mathbb{R}$. Consequently, $L_1 u \in \tilde{\mathcal{C}}_\varphi$.

Step 2: Continuity:

It is obvious from the linearity and boundedness of operator L_1 .

Step 3: Compactness:

Let $B \subset \tilde{\mathcal{C}}_\varphi$ be a bounded set, that is, there exists some $R > 0$ such that $\|u\|_\varphi \leq R$ for all $u \in B$.

Then,

$$\begin{aligned} \|L_1 u\|_\varphi &= \left\| \widetilde{L_1 u} \right\|_\infty = \left\| \frac{L_1 u}{\varphi} \right\|_\infty = \left\| \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} |k(\cdot, s) \eta(s)| u(s) \, ds \right\|_\infty \\ &\leq \|u\|_\varphi \left\| \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} |k(\cdot, s) \eta(s)| \varphi(s) \, ds \right\|_\infty \\ &\leq R \left\| \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} |k(\cdot, s) \eta(s)| \varphi(s) \, ds \right\|_\infty < \infty, \end{aligned} \quad (8.5.3)$$

and we have obtained an upper bound which does not depend on u . Therefore it is clear that the set $L_1(B)$ is uniformly bounded.

On the other hand, taking into account the upper bound found in (8.5.2), we have that if $t_1, t_2 \in \mathbb{R}$ are such that $|t_1 - t_2| < \delta$ then

$$\left| \widetilde{L_1 u}(t_1) - \widetilde{L_1 u}(t_2) \right| \leq \varepsilon \|u\|_\varphi \int_{-\infty}^{\infty} \omega_0(s) \varphi(s) \, ds \leq \varepsilon R \int_{-\infty}^{\infty} \omega_0(s) \varphi(s) \, ds,$$

and, since $\omega_0 \varphi \in L^1(\mathbb{R})$, we conclude that $L_1(B)$ is equicontinuous.

In conclusion, we derive, by application of Ascoli-Arzelà's Theorem (Theorem 8.3.2), that $L_1(B)$ is relatively compact in $\tilde{\mathcal{C}}_\varphi$ and therefore L_1 is a compact operator.

Step 4: L_1 maps P to $P \cap K_\alpha$:

Since L_1 has a positive integral kernel, it clearly maps P into P . Finally, it maps P into $P \cap K_\alpha$ as a direct consequence of hypotheses $(\tilde{\mathcal{C}}_4)$ and $(\tilde{\mathcal{C}}_5)$.

CASE II: $n \neq 0$:

We note that in this case we have the additional hypothesis that $k(\cdot, s) \eta(s)$ is nonnegative for all $s \in \mathbb{R}$. As a consequence, we will omit the absolute value in the definition of $L_1 u$.

As in Case I, we will divide the proof into four steps.

Step 1: L_1 maps $(\tilde{\mathcal{C}}_\varphi^n, \|\cdot\|_\varphi)$ to $(\tilde{\mathcal{C}}_\varphi^n, \|\cdot\|_\varphi)$:

Let $u \in \tilde{\mathcal{C}}_\varphi^n$.

Since $\frac{k(\cdot, s)\eta(s)}{\varphi(\cdot)}$ is integrable for every $s \in \mathbb{R}$, we can use Leibniz's Integral Rule to get

$$\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t) = \frac{\partial^j (L_1 u / \varphi)}{\partial t^j}(t) = \int_{-\infty}^{\infty} \frac{\partial^j (k/\varphi)}{\partial t^j}(t, s) \eta(s) u(s) \, ds.$$

On the other hand, from (\widetilde{C}_1) , given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\begin{aligned} & \left| \frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t_1) - \frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t_2) \right| \\ & \leq \int_{-\infty}^{\infty} \left| \frac{\partial^j (k/\varphi)}{\partial t^j}(t_1, s) \eta(s) - \frac{\partial^j (k/\varphi)}{\partial t^j}(t_2, s) \eta(s) \right| |u(s)| \, ds \quad (8.5.4) \\ & \leq \varepsilon \int_{-\infty}^{\infty} \omega_j(s) |u(s)| \, ds \leq \varepsilon \|u\|_{\varphi} \int_{-\infty}^{\infty} \omega_j(s) \varphi(s) \, ds. \end{aligned}$$

Since $\omega_j \varphi \in L^1(\mathbb{R})$, the previous expression is bounded from above by $\varepsilon \|u\|_{\varphi} c$ for some positive constant c . Hence, $\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}$ is continuous in \mathbb{R} for $j = 0, \dots, n$, that is, $\widetilde{L_1 u} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R})$.

Analogously to Case I, it can be proved that there exists $\lim_{t \rightarrow \pm\infty} \widetilde{L_1 u}(t)$ and, consequently, $L_1 u \in \widetilde{\mathcal{C}}_{\varphi}^n$.

Step 2: Continuity:

Again, it is obvious from the linearity and boundedness of operator L_1 .

Step 3: Compactness:

Let $B \subset \widetilde{\mathcal{C}}_{\varphi}^n$ be a bounded set, that is, there exists $R > 0$ such that $\|u\|_{\varphi} \leq R$ for all $u \in B$.

We will prove that $L_1(B)$ is uniformly bounded.

Using the General Leibniz's Rule (for differentiation), it is clear that

$$\frac{\partial^j \widetilde{L_1 u}}{\partial t^j} = \frac{\partial^j (L_1 u / \varphi)}{\partial t^j} = \sum_{l=0}^j \binom{j}{l} \frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}.$$

Moreover, from Leibniz's Integral Rule,

$$\frac{\partial^l L_1 u}{\partial t^l}(t) = \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) u(s) \, ds.$$

Thus,

$$\begin{aligned} \left\| \frac{\partial^j \widetilde{L_1 u}}{\partial t^j} \right\|_{\infty} &= \left\| \sum_{l=0}^j \binom{j}{l} \frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi} \right\|_{\infty} \leq \sum_{l=0}^j \binom{j}{l} \left\| \frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi} \right\|_{\infty} \\ &= \sum_{l=0}^j \binom{j}{l} \left\| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) u(s) \, ds \right\|_{\infty}. \end{aligned}$$

It is satisfied that

$$\begin{aligned} &\left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) u(s) \, ds \right| \\ &\leq \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \right| \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) \right| |u(s)| \, ds \\ &\leq R \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \right| \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(t, s) \eta(s) \right| \varphi(s) \, ds, \end{aligned}$$

and so, from two previous inequalities and taking into account condition (\widetilde{C}_2) , we deduce that

$$\left\| \frac{\partial^j \widetilde{L_1 u}}{\partial t^j} \right\|_{\infty} \leq R \sum_{l=0}^j \binom{j}{l} \left\| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} \left| \frac{\partial^l k}{\partial t^l}(\cdot, s) \eta(s) \right| \varphi(s) \, ds \right\|_{\infty} < \infty.$$

The rest of the proof is analogous to Case I but using equation (8.5.4) instead of (8.5.2).

Step 4: L_1 maps P to $P \cap K_{\alpha}$.

The proof is the same than in Case I. □

Theorem 8.5.3. *If (\widetilde{C}_1) , (\widetilde{C}_2) , (\widetilde{C}_6) , (\widetilde{C}_7) and (\widetilde{C}_8) hold, then operator L_2 is continuous, compact and maps P into $P \cap K_{\alpha}$.*

Proof. We will distinguish two different cases:

CASE I: $n = 0$:

Step 1: L_2 maps $(\widetilde{\mathcal{C}}_{\varphi}, \|\cdot\|_{\varphi})$ to $(\widetilde{\mathcal{C}}_{\varphi}, \|\cdot\|_{\varphi})$:

Let $u \in \widetilde{\mathcal{C}}_{\varphi}$.

Since $k(\cdot, s) \eta(s) \in \widetilde{\mathcal{C}}_{\varphi}$ for all $s \in \mathbb{R}$, it is clear that

$$\left(\frac{k(\cdot, s) \eta(s)}{\varphi(\cdot)} \right)^+ \equiv \frac{(k(\cdot, s) \eta(s))^+}{\varphi(\cdot)} \in \mathcal{C}(\mathbb{R}) \quad \text{for all } s \in \mathbb{R}.$$

Reasoning analogously to the proof for L_1 , from (\tilde{C}_1) , (ii), given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\left| \widetilde{L_2 u}(t_1) - \widetilde{L_2 u}(t_2) \right| \leq \varepsilon \|u\|_\varphi \int_A \omega_0(s) \varphi(s) \, ds \quad (8.5.5)$$

and, since $\omega_0 \varphi \in L^1(\mathbb{R})$, it can be deduced that $\widetilde{L_2 u}$ is continuous in \mathbb{R} .

Now we will see that there exists $\lim_{t \rightarrow \pm\infty} \widetilde{L_2 u}(t) \in \mathbb{R}$. We have that

$$\lim_{t \rightarrow \pm\infty} \widetilde{L_2 u}(t) = \lim_{t \rightarrow \pm\infty} \frac{L_2 u(t)}{\varphi(t)} = \lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_A (k(t, s) \eta(s))^+ u(s) \, ds.$$

Reasoning as before, since $k(\cdot, s) \eta(s) \in \tilde{\mathcal{C}}_\varphi$, then $(k(\cdot, s) \eta(s))^+ \in \tilde{\mathcal{C}}_\varphi$ and so, for all $s \in \mathbb{R}$, it is ensured the existence of

$$0 \leq \lim_{t \rightarrow \pm\infty} \frac{(k(t, s) \eta(s))^+}{\varphi(t)} \leq \lim_{t \rightarrow \pm\infty} \frac{|k(t, s) \eta(s)|}{\varphi(t)} = z_{(\pm)}(s) \in \mathbb{R}.$$

On the other hand,

$$\begin{aligned} \left| \frac{(k(t, s) \eta(s))^+}{\varphi(t)} u(s) \right| &\leq \left| \frac{|k(t, s) \eta(s)|}{\varphi(t)} |u(s)| \right| \leq M_0(s) |u(s)| = M_0(s) \frac{|u(s)|}{\varphi(s)} \varphi(s) \\ &\leq \|u\|_\varphi M_0(s) \varphi(s) \end{aligned}$$

for all $t \in \mathbb{R}$. From (\tilde{C}_2) , $M_0 \varphi \in L^1(\mathbb{R})$ and so $M_0 \varphi \in L^1(A)$. Thus, from Lebesgue's Dominated Convergence Theorem,

$$\lim_{t \rightarrow \pm\infty} \frac{1}{\varphi(t)} \int_A (k(t, s) \eta(s))^+ u(s) \, ds = \int_A \lim_{t \rightarrow \pm\infty} \frac{(k(t, s) \eta(s))^+}{\varphi(t)} u(s) \, ds,$$

and since

$$\begin{aligned} \left| \int_A \lim_{t \rightarrow \pm\infty} \frac{(k(t, s) \eta(s))^+}{\varphi(t)} u(s) \, ds \right| &\leq \int_A z_{(\pm)}(s) |u(s)| \, ds \\ &\leq \|u\|_\varphi \int_A z_{(\pm)}(s) \varphi(s) \, ds \in \mathbb{R}, \end{aligned}$$

and $z_{(\pm)} \varphi \in L^1(A)$, it can be concluded that there exists $\lim_{t \rightarrow \pm\infty} \frac{L_2 u(t)}{\varphi(t)}$ and, consequently, $L_2 u \in \tilde{\mathcal{C}}_\varphi$.

Step 2: Continuity:

It is obvious from the linearity and boundedness of operator L_2 .

Step 3: Compactness:

The proof is analogous to the one for operator L_1 (Theorem 8.5.2) by considering inequalities. We have the following inequalities

$$\begin{aligned} \|\widetilde{L_2 u}\|_\infty &= \left\| \frac{L_2 u}{\varphi} \right\|_\infty = \left\| \frac{1}{\varphi(\cdot)} \int_A (k(\cdot, s) \eta(s))^+ u(s) \, ds \right\|_\infty \\ &\leq \|u\|_\varphi \left\| \frac{1}{\varphi(\cdot)} \int_A (k(\cdot, s) \eta(s))^+ \varphi(s) \, ds \right\|_\infty \\ &\leq \|u\|_\varphi \left\| \frac{1}{\varphi(\cdot)} \int_A |k(\cdot, s) \eta(s)| \varphi(s) \, ds \right\|_\infty \\ &\leq \|u\|_\varphi \left\| \frac{1}{\varphi(\cdot)} \int_{-\infty}^{\infty} |k(\cdot, s) \eta(s)| \varphi(s) \, ds \right\|_\infty, \end{aligned}$$

and (8.5.5) instead of (8.5.3) and (8.5.2), respectively.

Step 4: L_2 maps P to $P \cap K_\alpha$:

Since L_2 has a positive integral kernel, it clearly maps P into P . Finally, it maps P into $P \cap K_\alpha$ as a direct consequence of hypothesis (\tilde{C}_7) and (\tilde{C}_8) .

CASE II: $n \neq 0$:

The proof is analogous to the one made for operator L_1 , with some small changes in the line of those introduced in Case I. \square

Analogously to the two previous theorems, it can be proved that operator T satisfies the following properties.

Theorem 8.5.4. *If (\tilde{C}_1) – (\tilde{C}_3) , (\tilde{C}_9) and (\tilde{C}_{10}) hold, then operator T is continuous, compact and maps K_α into K_α .*

Proof. The proof, except for the continuity, is completely analogous to previous theorems but using the following inequality

$$f(s, u(s)) = f\left(s, \frac{u(s)}{\varphi(s)} \varphi(s)\right) \leq \phi_{\|u\|_\varphi}(s) \varphi(s) \leq \|\phi_{\|u\|_\varphi}\|_\infty \varphi(s),$$

instead of $u(s) \leq \|u\|_\varphi \varphi(s)$.

Continuity:

Since T is not a linear operator, continuity can not be deduced from boundedness, contrary to what we did in previous theorems. Therefore, we shall prove that operator T is continuous in a different way.

Let $\{u_n\}_{n \in \mathbb{N}} \subset \tilde{C}_\varphi$ be a sequence which converges to u in \tilde{C}_φ . Then, there exists some $R \in \mathbb{R}$ such that $\|u_n\|_\varphi \leq R$ for all $n \in \mathbb{N}$ and so it holds that

$$f(s, u_n(s)) = f\left(s, \frac{u_n(s)}{\varphi(s)} \varphi(s)\right) \leq \phi_R(s) \varphi(s) \leq \|\phi_R\|_\infty \varphi(s),$$

where we have used condition (\tilde{C}_3) .

Moreover, from (\tilde{C}_3) , it holds that $f(s, u_n(s)) \rightarrow f(s, u(s))$ for a. e. $s \in \mathbb{R}$.

From (8.4.2), it is clear that, for all $t \in \mathbb{R}$ and $j \in \{0, \dots, n\}$,

$$\left| \frac{\partial^j \widetilde{T u_n}}{\partial t^j}(t) - \frac{\partial^j \widetilde{T u}}{\partial t^j}(t) \right| \leq \int_{-\infty}^{\infty} \left| \frac{\partial^j (k/\varphi)}{\partial t^j}(t, s) \eta(s) \right| |f(s, u_n(s)) - f(s, u(s))| \, ds$$

and, using (\tilde{C}_2) ,

$$\begin{aligned} \left| \frac{\partial^j \widetilde{T u_n}}{\partial t^j}(t) - \frac{\partial^j \widetilde{T u}}{\partial t^j}(t) \right| &\leq \int_{-\infty}^{\infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &\leq 2 \|\phi_R\|_{\infty} \int_{-\infty}^{\infty} M_j(s) \varphi(s) \, ds < \infty. \end{aligned}$$

Now we deduce, by application of Lebesgue's Dominated Convergence Theorem, that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \frac{\partial^j \widetilde{T u_n}}{\partial t^j} - \frac{\partial^j \widetilde{T u}}{\partial t^j} \right\|_{\infty} &\leq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &= \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} M_j(s) |f(s, u_n(s)) - f(s, u(s))| \, ds = 0. \end{aligned}$$

Therefore, we deduce that $T u_n \rightarrow T u$ in \tilde{C}_{φ}^n and so T is continuous. \square

The following theorem proves that the spectral radius (see Definition 1.3.2) of both operators L_1 and L_2 are positive and their related eigenfunctions have constant sign. This result is analogous to [79, Theorem 4.5] and is proven using the facts that the considered operators leave P invariant and that P is a total cone (see Definition 1.2.5), combined with Krein-Rutman Theorem (Theorem 1.3.3).

Theorem 8.5.5. *Assume that conditions (\tilde{C}_1) , (\tilde{C}_2) and (\tilde{C}_4) – (\tilde{C}_8) hold. Then, it holds that $r(L_1) > 0$ is an eigenvalue of L_1 with an eigenfunction in $P \setminus \{0\}$.*

Analogously, $r(L_2) > 0$ is an eigenvalue of L_2 with an eigenfunction in $P \setminus \{0\}$.

Proof. We will prove the result for L_1 . Consider $v \in P$ such that $v \equiv 1$ in A . Then, for $t \in A$,

$$\begin{aligned} L_1 v(t) &= \int_{-\infty}^{\infty} |k(t, s) \eta(s)| v(s) \, ds \geq \int_A |k(t, s) \eta(s)| v(s) \, ds \\ &= \int_A k(t, s) \eta(s) \, ds \geq \frac{1}{M}, \end{aligned}$$

with $1/\widetilde{M}$ given in (\tilde{C}_6) .

Then, there exists some open and bounded set B , with $A \subset B$ such that when $t \in B$,

$$\int_A |k(t, s) \eta(s)| \, ds \geq \frac{1}{2\widetilde{M}}.$$

Now, defining $u(t) = 1$ for $t \in A$ and $u(t) = 0$ when $t \notin B$, from Whitney's Extension Theorem (see [153, Theorem I]), u can be extended to \mathbb{R} (and this extension will be also denoted by u) as a function of class n . Moreover, from the proof of Whitney's Extension Theorem, it is possible to deduce that this extension will be nonnegative and upperly bounded by 1.

Finally, since $u(t) = 0$ when $t \notin B$ and B is a bounded set, then it is clear that $\lim_{t \rightarrow \pm\infty} u(t) = 0$ and $u \in \tilde{C}_\varphi^n$, with independence of the choice of φ .

Therefore, for $t \in B$, it holds that

$$\begin{aligned} L_1 u(t) &= \int_{-\infty}^{\infty} |k(t, s) \eta(s)| u(s) \, ds \geq \int_A |k(t, s) \eta(s)| u(s) \, ds \\ &= \int_A |k(t, s) \eta(s)| \, ds \geq \frac{1}{2\widetilde{M}} \geq \frac{1}{2\widetilde{M}} u(t), \end{aligned}$$

and for $t \notin B$,

$$L_1 u(t) = \int_{-\infty}^{\infty} |k(t, s) \eta(s)| u(s) \, ds \geq 0 = \frac{1}{2\widetilde{M}} u(t).$$

Thus, as a consequence of Theorem 1.3.6, we conclude that

$$r(L_1) \geq \frac{1}{2\widetilde{M}} > 0.$$

Finally, since P is a total cone and L_1 maps P into P , Krein-Rutman Theorem (Theorem 1.3.3) ensures that $r(L_1)$ is an eigenvalue with a related eigenfunction $\phi \in P \setminus \{0\}$. \square

Remark 8.5.6. As a consequence of Theorems 8.5.2 and 8.5.5, we know that the eigenfunctions mentioned above are in $P \cap K_\alpha$.

We will define now the following operator on $\mathcal{C}^n(A, \mathbb{R})$

$$\bar{L}u(t) := \int_A k(t, s) \eta(s) u(s) \, ds, \quad t \in A,$$

and consider the cone P_A of positive functions in $\mathcal{C}^n(A, \mathbb{R})$.

As with previous operators, we will prove that \bar{L} satisfies the following properties.

Theorem 8.5.7. Assume that conditions (\tilde{C}_1) , (\tilde{C}_2) and (\tilde{C}_6) – (\tilde{C}_8) hold. Then, the operator \bar{L} is compact and maps P_A into P_A .

Proof. Let $f \in \mathcal{C}^n(A, \mathbb{R})$ and $B \subset \mathbb{R}$ an open and bounded set such that $A \subset B$. Define now $g(t) = f(t)$ for $t \in A$ and $g(t) = 0$ for $t \in \mathbb{R} \setminus B$. Then, from Whitney's Extension Theorem (see [153, Theorem I]), g can be extended to \mathbb{R} as a function of class n , that is, there exists an extension of f to \mathbb{R} as a function of class n such that this extension vanishes for $t \in \mathbb{R} \setminus B$. Obviously, this extension of f belongs to $\mathcal{C}_\varphi^n(\mathbb{R})$.

Now, denote by i the function which maps a function in $\mathcal{C}^n(A, \mathbb{R})$ to the aforementioned extension in $\mathcal{C}_\varphi^n(\mathbb{R})$ and by π the map which takes every function in $\mathcal{C}_\varphi^n(\mathbb{R})$ to its restriction to the set A (which clearly belongs to $\mathcal{C}^n(A, \mathbb{R})$). We obtain the following diagram:

$$\begin{array}{ccc} \mathcal{C}_\varphi^n(\mathbb{R}) & \xrightarrow{L_2} & \mathcal{C}_\varphi^n(\mathbb{R}) \\ i \uparrow & & \downarrow \pi \\ \mathcal{C}^n(A, \mathbb{R}) & \xrightarrow{\bar{L}} & \mathcal{C}^n(A, \mathbb{R}) \end{array}$$

Let us show now that it is commutative. Consider $f \in \mathcal{C}^n(A, \mathbb{R})$. It holds that

$$\begin{aligned} (\pi \circ L_2 \circ i)(f)(t) &= \pi \left(\int_A (k(t, s) \eta(s))^+ i(f)(s) \, ds \right) \\ &= \pi \left(\int_A (k(t, s) \eta(s))^+ f(s) \, ds \right) \\ &= \int_A k(t, s) \eta(s) f(s) \, ds = \bar{L}(f)(t), \quad t \in A. \end{aligned}$$

Now, since L_2 is compact and both i and π are continuous, we deduce that \bar{L} is a compact operator.

Finally, from (\tilde{C}_6) it is clear that \bar{L} maps P_A into P_A . □

Remark 8.5.8. We point out that, in the previous proof, Whitney's extension theorem can be used as a consequence of the fact that A is a finite union of compact intervals.

Theorem 8.5.9. It holds that $r(\bar{L}) > 0$ and it is an eigenvalue of \bar{L} with an eigenfunction in P_A .

Proof. Let ψ be the eigenfunction related to L_2 whose existence is proved in Theorem 8.5.5. Then, if we consider its restriction to A , $\psi|_A$, it is clear that for $t \in A$

$$\bar{L}\psi|_A(t) = L_2\psi(t) = r(L_2)\psi(t) = r(L_2)\psi|_A(t),$$

and so from Theorems 1.3.6 and 8.5.5, we deduce that $r(\bar{L}) \geq r(L_2) > 0$. \square

We define the following numbers in the extended real line:

$$f^0 = \overline{\lim}_{x \rightarrow 0} \frac{\sup_{t \in \mathbb{R}} \frac{f(t, x \varphi(t))}{\varphi(t)}}{|x|}, \quad f_0 = \underline{\lim}_{x \rightarrow 0} \frac{\inf_{t \in A} \frac{f(t, x \varphi(t))}{\varphi(t)}}{|x|},$$

$$f^\infty = \overline{\lim}_{|x| \rightarrow +\infty} \frac{\sup_{t \in \mathbb{R}} \frac{f(t, x \varphi(t))}{\varphi(t)}}{|x|}, \quad f_\infty = \underline{\lim}_{|x| \rightarrow +\infty} \frac{\inf_{t \in A} \frac{f(t, x \varphi(t))}{\varphi(t)}}{|x|}.$$

Next, we will give a result in which we will prove that, under suitable conditions, the index of some subsets is 1 or 0. Before that, we shall give the following definition that will be implicitly used in Theorem 8.5.11.

Definition 8.5.10. Let X, Y, Z be topological spaces, Y Hausdorff. Let $f : X \rightarrow Y$, $g : X \rightarrow Z$. Let $z_0 \in g(X)'$. We say that L is the limit of f when $g(x)$ tends to z_0 if for every neighborhood N_Y of L there exists a neighborhood N_Z of z_0 such that $f(g^{-1}(N_Z \setminus \{z_0\})) \subset N_Y$. We write

$$\lim_{g(x) \rightarrow z_0} f(x) = L.$$

A particular case of this definition would be the notion of limit in the case of the topology occurring when studying Stieltjes derivatives with respect to a function g (cf. [59, 120]).

Now, in order to prove the following theorem, we adapt some of the proofs of [151, Theorems 3.2-3.5] to this new context. In particular, to prove that the index of some subsets of a cone is 1 or 0, we will use the sufficient conditions given in Lemma 1.2.7.

Theorem 8.5.11. Assume that hypotheses (\tilde{C}_1) – (\tilde{C}_{10}) hold. Assume also that there exists $\beta : \mathcal{C}_\varphi^n \rightarrow [0, \infty)$ such that

$$\lim_{\beta(u) \rightarrow 0} \|u\|_\varphi = 0, \quad \lim_{\beta(u) \rightarrow +\infty} \|u\|_\varphi = +\infty,$$

and

$$\beta(u) \neq 0 \Rightarrow u \neq 0.$$

Consider $K_\alpha^{\beta, \rho} := \{u \in K_\alpha : \beta(u) < \rho\}$. Then, the following assertions hold:

- (1) If $0 \leq f^0 < \mu(L_1)$, then there exists $\rho_0 > 0$ such that $i_{K_\alpha}(T, K_\alpha^{\beta, \rho}) = 1$ for each $\rho \in (0, \rho_0]$.

- (2) If $0 \leq f^\infty < \mu(L_1)$, then there exists $R_0 > 0$ such that $i_{K_\alpha}(T, K_\alpha^{\beta, R}) = 1$ for each $R > R_0$.
- (3) If $\mu(L_2) < f_0 \leq \infty$, then there exists $\rho_0 > 0$ such that $K_\alpha^{\beta, \rho}$ is bounded and $i_{K_\alpha}(T, K_\alpha^{\beta, \rho}) = 0$ for each $\rho \in (0, \rho_0]$.
- (4) If $\mu(L_2) < f_\infty \leq \infty$ and there exists $R_1 > 0$ such that $K_\alpha^{\beta, R}$ is bounded for all $R \geq R_1$, then $i_{K_\alpha}(T, K_\alpha^{\beta, R}) = 0$ for each $R \geq R_1$.

We recall that μ denotes the characteristic value of a bounded linear operator (see Definition 1.3.2).

Proof. (1) Let $\tau > 0$ be such that $f^0 < \mu(L_1) - \tau =: \xi$. Then there exists $\tilde{\rho}_0 \in (0, 1)$ such that, for all $x \in [-\tilde{\rho}_0, \tilde{\rho}_0]$ and almost every $t \in \mathbb{R}$, we have

$$f(t, x \varphi(t)) \leq \xi |x| \varphi(t).$$

Also, since $\lim_{\beta(u) \rightarrow 0} \|u\|_\varphi = 0$, there is $\rho_0 < \tilde{\rho}_0$ such that

$$\|u\|_\varphi < \tilde{\rho}_0 \text{ for } u \in \overline{K_\alpha^{\beta, \rho_0}}.$$

Let $\rho \in (0, \rho_0]$. We will prove that $Tu \neq \lambda u$ for $u \in \partial K_\alpha^{\beta, \rho}$ and $\lambda \geq 1$, which implies that $i_K(T, K_\alpha^{\beta, \rho}) = 1$. In fact, if we assume otherwise, then there exist $u \in \partial K_\alpha^{\beta, \rho}$ (that is, $\beta(u) = \rho$ and therefore, $u \neq 0$) and $\lambda \geq 1$ such that $\lambda u = Tu$. Therefore, for $t \in \mathbb{R}$,

$$\begin{aligned} |u(t)| &\leq \lambda |u(t)| = |Tu(t)| = \left| \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds \right| \\ &\leq \int_{-\infty}^{\infty} |k(t, s) \eta(s)| f\left(s, \frac{u(s)}{\varphi(s)} \varphi(s)\right) \, ds \leq \xi \int_{-\infty}^{\infty} |k(t, s) \eta(s)| |u(s)| \, ds \\ &= \xi (L_1 |u|)(t). \end{aligned}$$

We conclude that $|u| \leq \xi L_1 |u|$. Thus, as L_1 is a nondecreasing operator, iterating, we have that

$$|u| \leq \xi L_1 |u| \leq \xi L_1 (\xi L_1 |u|) = \xi^2 L_1^2 |u| \leq \cdots \leq \xi^n L_1^n |u|.$$

That is,

$$\|u\|_\varphi \leq \xi^n \|L_1^n |u|\|_\varphi$$

and, hence,

$$1 \leq \xi^n \frac{\|L_1^n |u|\|_\varphi}{\|u\|_\varphi} \leq \xi^n \|L_1^n\|_\varphi,$$

where $\|L_1^n\|_\varphi$ denotes the norm of the operator, namely

$$\|L_1^n\|_\varphi = \sup_{u \neq 0} \frac{\|L_1^n u\|_\varphi}{\|u\|_\varphi}.$$

Taking the n -th square root and the limit when $n \rightarrow \infty$, we get

$$1 \leq \xi (\|L_1^n\|_\varphi)^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} \xi r(L_1),$$

which is a contradiction.

(2) Let $\tau \in \mathbb{R}^+$ such that $f^\infty < \mu(L_1) - \tau =: \xi$. Then there exists $R_1 > 0$ such that for every $|x| \geq R_1$ and almost every $t \in \mathbb{R}$

$$f(t, x \varphi(t)) \leq \xi |x| \varphi(t).$$

Also, by (\tilde{C}_3) there exists $\phi_{R_1} \in L^\infty(\mathbb{R})$ such that

$$\frac{f(t, x \varphi(t))}{\varphi(t)} \leq \phi_{R_1}(t),$$

for all $x \in [-R_1, R_1]$ and a. e. $t \in \mathbb{R}$. Hence,

$$f(t, x \varphi(t)) \leq \xi |x| \varphi(t) + \varphi(t) \phi_{R_1}(t) \text{ for all } x \in \mathbb{R} \text{ and a. e. } t \in \mathbb{R}. \quad (8.5.6)$$

Moreover, since $\xi < \frac{1}{r(L_1)}$ we deduce that

$$r(\xi L_1) = \xi r(L_1) < 1.$$

Thus, if we denote by Id the identity operator, since ξL_1 has spectral radius less than one, $\text{Id} - \xi L_1$ is invertible. Furthermore, by the Neumann series expression,

$$(\text{Id} - \xi L_1)^{-1} = \sum_{k=0}^{\infty} (\xi L_1)^k$$

and therefore, $(\text{Id} - \xi L_1)^{-1}$ maps P into $P \cap K_\alpha$, since L_1 does.

Since $\phi_{R_1} \in L^\infty(\mathbb{R})$,

$$C(t) := \int_{-\infty}^{\infty} |k(t, s) \eta(s)| \varphi(s) \phi_{R_1}(s) \, ds \leq \|\phi_{R_1}\|_\infty \int_{-\infty}^{\infty} |k(t, s) \eta(s)| \varphi(s) \, ds,$$

and so, from (\tilde{C}_2) , it is clear that $C \in \tilde{\mathcal{C}}_\varphi^n$. Furthermore, since $C(t) \geq 0$ for all $t \in \mathbb{R}$, $C \in P$. Therefore $(\text{Id} - \xi L_1)^{-1} C \in P \cap K_\alpha$ and

$$R_0 := \|(\text{Id} - \xi L_1)^{-1} C\|_\varphi < +\infty.$$

Because $\lim_{\beta(u) \rightarrow +\infty} \|u\|_\varphi = +\infty$, there exists $R_2 > R_0$ such that

$$\|u\|_\varphi > R_0 \text{ for every } u \in \partial K_\alpha^{\beta, R} \text{ with } R > R_2.$$

Now we prove that for each $R > R_2$, $Tu \neq \lambda u$ for all $u \in \partial K_\alpha^{\beta, R}$ and $\lambda \geq 1$, which implies, from Lemma 1.2.7, that $i_K(T, K_\alpha^{\beta, R}) = 1$. Assume, otherwise, that there exist $u \in \partial K_\alpha^{\beta, R}$ and $\lambda \geq 1$ such that $\lambda u = Tu$. Taking into account the inequality (8.5.6), we have, for $t \in \mathbb{R}$,

$$\begin{aligned} |u(t)| &\leq \lambda |u(t)| = |Tu(t)| = \left| \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds \right| \\ &\leq \int_{-\infty}^{\infty} |k(t, s) \eta(s)| f\left(s, \frac{u(s)}{\varphi(s)} \varphi(s)\right) \, ds \\ &\leq \int_{-\infty}^{\infty} |k(t, s) \eta(s)| \left[\xi \left| \frac{u(s)}{\varphi(s)} \right| \varphi(s) + \varphi(s) \phi_{R_1}(s) \right] \, ds \\ &\leq \xi \int_{-\infty}^{\infty} |k(t, s) \eta(s)| |u(s)| \, ds + C(t) = \xi L_1 |u|(t) + C(t), \end{aligned}$$

which implies that

$$(\text{Id} - \xi L_1) |u|(t) \leq C(t).$$

Since $(\text{Id} - \xi L_1)^{-1}$ is nonnegative (and, therefore, nondecreasing), we have that

$$|u(t)| \leq (\text{Id} - \xi L_1)^{-1} C(t)$$

and, consequently,

$$\|u\|_\varphi \leq \|(\text{Id} - \xi L_1)^{-1} C\|_\varphi = R_0.$$

Therefore, we have that $\|u\|_\varphi \leq R_0$, which is a contradiction.

(3) There exists $\rho_0 > 0$ such that for all $x \in (0, \rho_0]$ and all $t \in A$ we have that

$$f(t, x \varphi(t)) \geq \mu(L_2) x \varphi(t).$$

Since $\lim_{\beta(u) \rightarrow 0} \|u\|_\varphi = 0$, there exists $\rho_1 \in (0, \rho_0]$ such that

$$\|u\|_\varphi < \rho_0 \text{ for every } u \in K_\alpha^{\beta, \rho}, \rho \in (0, \rho_1].$$

Let $\rho \in (0, \rho_1]$ be fixed. Let us prove that $u \neq Tu + \lambda \varphi_1$ for all u in $\partial K_\alpha^{\beta, \rho}$ and $\lambda \geq 0$, where $\varphi_1 \in K_\alpha \cap P$ is the eigenfunction of L_2 with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/\mu(L_2)$, whose existence is proved in Theorem 8.5.5. From Lemma 1.2.7, this implies that $i_K(T, K_\alpha^{\beta, \rho}) = 0$.

Assume, on the contrary, that there exist $u \in \partial K_{\alpha}^{\beta, \rho}$ and $\lambda \geq 0$ such that $u = Tu + \lambda \varphi_1$. We distinguish two cases.

Firstly, we discuss the case $\lambda > 0$. We have, for $t \in A$ in the conditions of (\tilde{C}_6) , that

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds + \lambda \varphi_1(t) \\ &\geq \int_A k(t, s) \eta(s) f\left(s, \frac{u(s)}{\varphi(s)} \varphi(s)\right) \, ds + \lambda \varphi_1(t) \\ &\geq \mu(L_2) \int_A k(t, s) \eta(s) u(s) \, ds + \lambda \varphi_1(t) = \mu(L_2) L_2 u(t) + \lambda \varphi_1(t). \end{aligned}$$

Note that the equality $u = Tu + \lambda \varphi_1$ implies that $u(t) \geq 0$ for $t \in A$. Therefore, $\mu(L_2) L_2 u(t) \geq 0$ for $t \in A$ and we deduce, from previous inequalities, that

$$u(t) \geq \lambda \varphi_1(t) \quad \text{for } t \in A.$$

Hence, for $t \in A$,

$$L_2 u(t) \geq \lambda L_2 \varphi_1(t) = \frac{\lambda}{\mu(L_2)} \varphi_1(t),$$

in such a way that we obtain

$$u(t) \geq \mu(L_2) L_2 u(t) + \lambda \varphi_1(t) \geq 2\lambda \varphi_1(t), \quad \text{for } t \in A.$$

By iteration, we deduce that, for $t \in A$, we get

$$u(t) \geq n \lambda \varphi_1(t) \quad \text{for every } n \in \mathbb{N},$$

a contradiction because $u(t)$ is finite and $\varphi_1|_A \not\equiv 0$.

Now we consider the case $\lambda = 0$. Let $\varepsilon > 0$ be such that for all $x \in (0, \rho_0]$ and almost every $t \in A$ we have

$$f(t, x \varphi(t)) \geq (\mu(L_2) + \varepsilon) x \varphi(t).$$

We have, for $t \in A$,

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds \geq \int_A (k(t, s) \eta(s))^+ f(s, u(s)) \, ds \\ &\geq (\mu(L_2) + \varepsilon) \int_A (k(t, s) \eta(s))^+ u(s) \, ds = (\mu(L_2) + \varepsilon) L_2 u(t). \end{aligned}$$

From previous expression together with (\tilde{C}_6) , it is immediately deduced that $u(t) > 0$ for $t \in A$.

Since $L_2\varphi_1(t) = r(L_2)\varphi_1(t)$ for $t \in \mathbb{R}$, we have, for $t \in A$,

$$\bar{L}\varphi_1(t) = L_2\varphi_1(t) = r(L_2)\varphi_1(t),$$

and we obtain $r(\bar{L}) \geq r(L_2)$.

On the other hand, we have, for $t \in A$,

$$u(t) \geq (\mu(L_2) + \varepsilon)L_2u(t) = (\mu(L_2) + \varepsilon)\bar{L}u(t),$$

where $u(t) > 0$. Thus, using Theorem 1.3.7, we have

$$r(\bar{L}) \leq \frac{1}{\mu(L_2) + \varepsilon}$$

and therefore

$$r(L_2) \leq \frac{1}{\mu(L_2) + \varepsilon}.$$

This gives $\mu(L_2) + \varepsilon \leq \mu(L_2)$, which is a contradiction.

(4) Let $R_1 > 0$ be such that

$$f(t, x\varphi(t)) > \mu(L_2)x\varphi(t)$$

for all $x > R_1$ and all $t \in A$.

Moreover, since $\lim_{\beta(u) \rightarrow +\infty} \|u\|_\varphi = +\infty$, there exists R_2 such that

$$\|u\|_\varphi > R_1 \text{ for every } u \in \partial K_\alpha^{\beta, R}, \text{ for } R > R_2.$$

Let $R \geq R_2$. Now, proceeding as in the proof of Statement (3), it is easy to prove that $u \neq Tu + \lambda\varphi_1$ for all u in $\partial K_\alpha^{\beta, R}$ and $\lambda \geq 0$, which implies that $i_K(T, K_\alpha^{\beta, R}) = 0$. \square

The next theorem, following the line of [152], applies the index results in Theorem 8.5.11 in order to prove the existence of nontrivial solutions for the equation (8.5.1).

Theorem 8.5.12. *Assume that conditions (\tilde{C}_1) – (\tilde{C}_{10}) hold. Suppose also that one of the following conditions is satisfied*

$$(T_1) \quad 0 \leq f^0 < \mu(L_1) \text{ and } \mu(L_2) < f_\infty \leq \infty.$$

$$(T_2) \quad 0 \leq f^\infty < \mu(L_1) \text{ and } \mu(L_2) < f_0 \leq \infty.$$

Then the integral equation (8.5.1) has at least one nontrivial solution in K_α .

Proof. We will prove (T_1) , being (T_2) analogous.

Take $\beta(u) = \|u\|_\varphi$. Clearly β is in the conditions of Theorem 8.5.11. Then, the existence of $\rho_0 > 0$ and $R_1 > 0$ such that $i_{K_\alpha}(T, K_\alpha^{\beta, \rho}) = 1$ for each $\rho \in (0, \rho_0]$ and $i_{K_\alpha}(T, K_\alpha^{\beta, R}) = 0$ for each $R \geq R_1$ is ensured.

Therefore, if we choose $\rho \leq \rho_0$ and $R \geq R_1$ such that $\rho < R$, we have that $K_\alpha^{\beta, \rho} \subset K_\alpha^{\beta, R}$, and from (3) and (4) in Lemma 1.2.7 we deduce that T has a fixed point in $K_\alpha^{\beta, R} \setminus \overline{K_\alpha^{\beta, \rho}}$. \square

The following lemma establishes some relations between the characteristic values of some of the considered operators.

Lemma 8.5.13. *It holds that $\widetilde{M}(A) \geq \mu(L_2) \geq \mu(L_1)$.*

Proof. First, we prove that $\mu(L_2) \geq \mu(L_1)$. Let $\phi \in \widetilde{\mathcal{C}}_\varphi^n \setminus \{0\}$ be an eigenfunction of L_1 related to the eigenvalue $r(L_1)$. We have that

$$\begin{aligned} r(L_1) \phi(t) &= L_1 \phi(t) = \int_{-\infty}^{\infty} |k(t, s) \eta(s)| \phi(s) \, ds \geq \int_A |k(t, s) \eta(s)| \phi(s) \, ds \\ &\geq \int_A (k(t, s) \eta(s))^+ \phi(s) \, ds = L_2 \phi(t). \end{aligned}$$

Thus, Theorem 1.3.7 implies that $r(L_2) \leq r(L_1)$ or, equivalently, $\mu(L_2) \geq \mu(L_1)$.

Now we prove $\widetilde{M}(A) \geq \mu(L_2)$. Let $\phi \in P \cap K_\alpha$ be a corresponding eigenfunction of norm 1 of $1/\mu(L_2)$ for the operator L_2 , that is $\phi = \mu(L_2) L_2(\phi)$ and $\|\phi\| = 1$. Then, for $t \in A$, we have that

$$\phi(t) = \mu(L_2) \int_A k(t, s) \eta(s) \phi(s) \, ds \geq \mu(L_2) \min_{t \in A} \phi(t) \int_A k(t, s) \eta(s) \, ds.$$

Taking the minimum over A , we obtain

$$\min_{t \in A} \phi(t) \geq \mu(L_2) \min_{t \in A} \phi(t) \frac{1}{\widetilde{M}(A)},$$

that is, $\widetilde{M}(A) \geq \mu(L_2)$. \square

Remark 8.5.14. *We note that all the previous results could also be formulated for the spaces $\widetilde{\mathcal{C}}_\varphi^n([a, +\infty))$ or $\widetilde{\mathcal{C}}_\varphi^n((-\infty, a])$ for any $a \in \mathbb{R}$.*

8.5.1. An Example

Consider the problem

$$Tu(t) = \int_{-\infty}^{\infty} e^{-\frac{|s|}{2}} \sin t \sqrt{|u(s)|} \sin^2 s \, ds,$$

that is, $k(t, s) = e^{-\frac{|s|}{2}} \sin t$, $\eta \equiv 1$ and $f(s, y) = \sqrt{|y|} \sin^2 s$.

We will take

$$\varphi(t) = |t|,$$

and

$$\alpha(u) = \min_{t \in [\frac{\pi}{4}, \frac{3\pi}{4}]} u(t) - \frac{\sqrt{2}}{2} \|u\|_{\infty}.$$

We will verify that conditions (\tilde{C}_1) – (\tilde{C}_{10}) are satisfied for the case $n = 0$:

(\tilde{C}_1) First of all, since $k(\cdot, s) \in \mathcal{C}(\mathbb{R})$ and there exist

$$\lim_{t \rightarrow \pm\infty} \frac{k(t, s)}{\varphi(t)} = \lim_{t \rightarrow \pm\infty} \frac{e^{-\frac{|s|}{2}} \sin t}{|t|} = 0,$$

it is clear that $k(\cdot, s) \in \tilde{\mathcal{C}}_{\varphi}$ for all $s \in \mathbb{R}$.

Moreover, for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|t_1 - t_2| < \delta$,

(i)

$$\left| \frac{k(t_1, s)}{\varphi(t_1)} - \frac{k(t_2, s)}{\varphi(t_2)} \right| = \left| \frac{e^{-\frac{|s|}{2}} \sin t_1}{|t_1|} - \frac{e^{-\frac{|s|}{2}} \sin t_2}{|t_2|} \right| \leq \varepsilon e^{-\frac{|s|}{2}},$$

and

(ii)

$$\left| \frac{(k(t_1, s))^+}{\varphi(t_1)} - \frac{(k(t_2, s))^+}{\varphi(t_2)} \right| = \left| \frac{e^{-\frac{|s|}{2}} (\sin t_1)^+}{|t_1|} - \frac{e^{-\frac{|s|}{2}} (\sin t_2)^+}{|t_2|} \right| \leq \varepsilon e^{-\frac{|s|}{2}},$$

so we will take $\omega_0(s) = e^{-\frac{|s|}{2}}$.

(\tilde{C}_2) Clearly, it holds that $\omega_0 \varphi \in L^1(\mathbb{R})$. Also,

$$\frac{1}{\varphi(t)} \int_{-\infty}^{\infty} |k(t, s)| \varphi(s) \, ds = \frac{|\sin t|}{|t|} \int_{-\infty}^{\infty} e^{-\frac{|s|}{2}} |s| \, ds = 8 \frac{|\sin t|}{|t|} \in L^\infty(\mathbb{R}).$$

Moreover, in this case

$$z_{(\pm)}(s) = \lim_{t \rightarrow \pm\infty} \frac{e^{-\frac{|s|}{2}} |\sin t|}{|t|} = 0,$$

$$M_0(s) = \sup_{t \in \mathbb{R}} \frac{e^{-\frac{|s|}{2}} |\sin t|}{|t|} = e^{-\frac{|s|}{2}},$$

and it holds that $z_{(\pm)} \varphi, M_0 \varphi \in L^1(\mathbb{R})$.

(\tilde{C}_3) It is clear that $f(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}$ and $f(t, \cdot)$ is continuous for a.e. $t \in \mathbb{R}$. Finally, for each $r > 0$, there exists

$$\phi_r(t) = \frac{\sqrt{r} \sin^2 t}{\sqrt{|t|}} \in L^\infty(\mathbb{R})$$

such that

$$\frac{f(t, x \varphi(t))}{\varphi(t)} = \frac{\sqrt{|x t|} \sin^2 t}{|t|} \leq \phi_r(t),$$

for all $x \in [-r, r]$ and a.e. $t \in \mathbb{R}$.

(\tilde{C}_4) In this case,

$$\begin{aligned} \alpha(|k(\cdot, s)|) &= \min_{t \in [\frac{\pi}{4}, \frac{3\pi}{4}]} |k(t, s)| - \frac{\sqrt{2}}{2} \|k(\cdot, s)\|_\infty \\ &= e^{-\frac{|s|}{2}} \min_{t \in [\frac{\pi}{4}, \frac{3\pi}{4}]} |\sin t| - \frac{\sqrt{2}}{2} e^{-\frac{|s|}{2}} = 0. \end{aligned}$$

(\tilde{C}_5) It is clear that $\alpha(|k(\cdot, s)|) \varphi(s) \in L^1(\mathbb{R})$. Moreover, for all $u \in P$, it holds that

$$\begin{aligned} \alpha(L_1 u) &= \min_{t \in [\frac{\pi}{4}, \frac{3\pi}{4}]} \int_{-\infty}^{\infty} |k(t, s)| u(s) \, ds - \frac{\sqrt{2}}{2} \left\| \int_{-\infty}^{\infty} |k(\cdot, s)| u(s) \, ds \right\|_\infty \\ &\geq \int_{-\infty}^{\infty} \min_{t \in [\frac{\pi}{4}, \frac{3\pi}{4}]} |k(t, s)| u(s) \, ds - \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} \|k(\cdot, s)\|_\infty u(s) \, ds \\ &= \int_{-\infty}^{\infty} \alpha(|k(\cdot, s)|) u(s) \, ds. \end{aligned}$$

(\tilde{C}_6) We can take $A = [\frac{\pi}{4}, \frac{3\pi}{4}]$. For such A , we obtain

$$\begin{aligned} \frac{1}{\widetilde{M}(A)} &= \inf_{t \in A} \int_A e^{-\frac{|s|}{2}} \sin t \, ds = \inf_{t \in A} \left\{ 2 e^{-\frac{3\pi}{8}} \left(-1 + e^{\frac{\pi}{4}} \right) \sin t \right\} \\ &= \sqrt{2} e^{-\frac{3\pi}{8}} \left(-1 + e^{\frac{\pi}{4}} \right) > 0. \end{aligned}$$

(\tilde{C}_7) It is analogous to (\tilde{C}_4). The same occurs to (\tilde{C}_9).

(\tilde{C}_8) It is analogous to (\tilde{C}_5). The same occurs to (\tilde{C}_{10}).

Finally, we obtain the following values for the limits f^∞ and f_0 :

$$f^\infty = \lim_{|x| \rightarrow +\infty} \frac{\sup_{t \in \mathbb{R}} \frac{\sqrt{|x|} \sin^2 t}{\sqrt{|t|}}}{|x|} \leq \lim_{|x| \rightarrow +\infty} \frac{\sqrt{|x|}}{|x|} = 0,$$

and so $f^\infty = 0$.

Analogously,

$$f_0 = \lim_{|x| \rightarrow 0} \frac{\inf_{t \in A} \frac{\sqrt{|x|} \sin^2 t}{\sqrt{|t|}}}{|x|} = \lim_{|x| \rightarrow 0} \frac{\sqrt{|x|}}{\sqrt{3\pi}|x|} = +\infty.$$

On the other hand, since both $r(L_1)$ and $r(L_2)$ are positive (as it has been proved in Theorem 8.5.5), it holds that $\mu(L_1) > 0$ and $\mu(L_2) < +\infty$.

Thus, from (T_2) in Theorem 8.5.12, we deduce that our problem has at least a nontrivial solution in $K_\alpha \subset \tilde{C}_\varphi$.

Remark 8.5.15. *Note that, as it has been indicated before, the results in Section 8.4 are not applicable to this problem as the nonlinearity f does not satisfy condition (C_2).*

Chapter 9

On Unbounded Solutions of Singular Initial Value Problems with ϕ -Laplacian

In this chapter, we will study the existence of unbounded solutions of a singular nonlinear initial value problem with a ϕ -Laplacian.

The same problem has already been considered in [13], where the authors discussed the existence and properties of bounded solutions. Here we will focus our attention on unbounded solutions of the problem and provide sufficient conditions for their existence. This way, this chapter completes the results obtained in [13].

Contrary to previous chapters, since the problem is singular, it is not possible to construct an equivalent integral problem with a kernel given by a related Green's function. As a consequence, the techniques developed in this chapter are completely different to the ones used previously in this Thesis.

All the results in this chapter are collected in [131].

9.1. Introduction

The aim of this chapter is to analyse the singular nonlinear equation

$$(p(t) \phi(u'(t)))' + p(t) f(\phi(u(t))) = 0, \quad t > 0, \quad (9.1.1)$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad (9.1.2)$$

where $u_0 \in [L_0, L]$.

A special case of equation (9.1.1) with $\phi(u) \equiv u$ and $p(t) = t^{n-1}$, $n \in \mathbb{N}$, $n \geq 2$,

$$(t^{n-1} u'(t))' + t^{n-1} f(u(t)) = 0, \quad t > 0,$$

arises in many areas. For example, in the study of phase transition of Van der Waals fluids ([60]), in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population ([55]), in the homogeneous nucleation theory ([1]), in the relativistic cosmology for description of particles which

can be treated as domains in the universe ([100]), or in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces ([47]).

The above nonlinear equation was replaced with its abstract and more general form

$$(p(t) u'(t))' + q(t) f(u(t)) = 0, \quad t > 0,$$

which was investigated for $p = q$ in [122, 123, 125–128] and for $p \neq q$ in [12, 14, 130, 144]. Other problems without ϕ -Laplacian close to (9.1.1)–(9.1.2) can be found in [2, 8, 10, 90–92] and those with ϕ -Laplacian in [49, 82, 107, 108, 137].

Before precisising what the main objectives of this chapter are, we need to define what we understand by solution of problem (9.1.1)–(9.1.2).

Definition 9.1.1. *Let $[0, b) \subset [0, \infty)$ be a maximal interval such that a function $u \in C^1([0, b))$ with $\phi(u') \in C^1((0, b))$ satisfies equation (9.1.1) for every $t \in (0, b)$ and let u satisfy the initial conditions (9.1.2). Then u is called a solution of problem on $[0, b)$.*

If u is a solution of problem (9.1.1)–(9.1.2) on $[0, \infty)$, then u is called a solution of problem (9.1.1)–(9.1.2).

In particular, following the line of [13], we will distinguish three different types of solutions.

Definition 9.1.2. *Consider a solution of problem (9.1.1)–(9.1.2) with $u_0 \in (L_0, L)$ and denote*

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}.$$

If $u_{\sup} = L$, then u is called a homoclinic solution of problem (9.1.1)–(9.1.2).

If $u_{\sup} < L$, then u is called a damped solution of problem (9.1.1)–(9.1.2).

Solutions from Definition 9.1.2 are bounded. Therefore, we are mostly interested in another type of solutions specified in the next definition.

Definition 9.1.3. *Let u be a solution of problem (9.1.1)–(9.1.2) on $[0, b)$, where $b \in (0, \infty]$. If there exists some $c \in (0, b)$ such that*

$$u(c) = L, \quad u'(c) > 0, \tag{9.1.3}$$

then u is called an escape solution of problem (9.1.1)–(9.1.2) on $[0, b)$.

The three considered types of solutions can be seen in Figure 9.1.1.

As we have mentioned before, analytical properties of the solutions of problem (9.1.1)–(9.1.2) with a ϕ -Laplacian have been already studied in [13], with a focus on

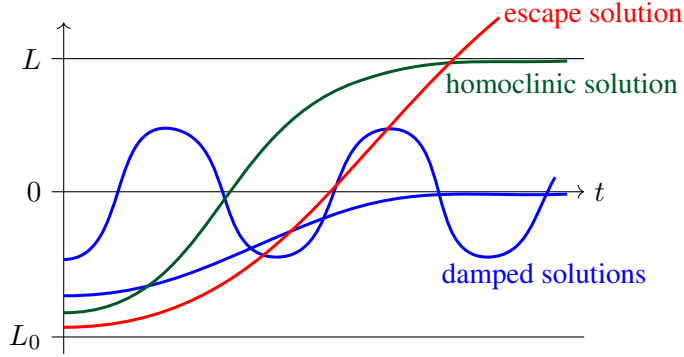


Figure 9.1.1: Types of solutions of problem (9.1.1)–(9.1.2).

the existence of bounded solutions on $[0, \infty)$. In particular, the existence of damped solutions was proved for certain values of u_0 .

Some results derived in [13] will also be useful here when the existence and properties of unbounded solutions are of interest. Therefore, we will recapitulate them in Section 9.2.

The goal of this chapter is to find conditions which guarantee the existence of escape solutions of problem (9.1.1)–(9.1.2) which are unbounded.

Note that the analysis of problem (9.1.1)–(9.1.2) with a general ϕ -Laplacian includes, for example, $\phi(x) = |x|^\alpha \text{sign } x$, for $\alpha > 1$. Let us emphasize that in this case, $\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \text{sign } x$ is not locally Lipschitz continuous. Since ϕ^{-1} is present in the operator form of (9.1.1)–(9.1.2), namely

$$u(t) = u_0 + \int_0^t \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau))) d\tau \right) ds, \quad t \geq 0,$$

the standard technique based on the Lipschitz property is not applicable here and another approach needs to be developed.

Therefore, we will distinguish two cases:

- In the first case, where functions ϕ^{-1} and f are Lipschitz continuous, the uniqueness of solution of problem (9.1.1)–(9.1.2) is guaranteed. In this case, we will obtain a sequence of escape solutions with different initial values.
- In the second case, functions ϕ^{-1} and f do not have to be Lipschitz continuous. The lack of uniqueness causes difficulties and therefore is more challenging. The problems are overcome by means of the lower and upper solutions method. Also here sufficient conditions for the existence of escape solutions are derived.

However, contrary to previous case, it may occur in this one that all the escape solutions have the same initial value L_0 .

Moreover, since, in general, an escape solution does not need to be unbounded, criteria for an escape solution to tend to infinity will be derived. In this manner, we will obtain new existence results for unbounded solutions of problem (9.1.1)–(9.1.2).

This chapter is organized in the following way: Preliminary results for an auxiliary problem with a bounded nonlinearity are stated in Section 9.2. Auxiliary lemmas necessary for proofs of the existence of escape solutions of the auxiliary problem are given in Section 9.3. The existence of escape solutions of this problem is further discussed in Section 9.4. Namely, the first existence result in Section 9.4 is derived by an approach based on the Lipschitz property. The other case without the Lipschitz condition is studied by means of the lower and upper solutions method. In Section 9.5, several criteria for escape solutions of the original problem to be unbounded are proved. The main results about the existence of unbounded solutions with examples are given in Section 9.6.

9.2. Preliminaries

Throughout this chapter, we will make the following basic assumptions:

- $\phi \in C^1(\mathbb{R})$ is a Laplacian, that is,

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0 \quad \text{and} \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}). \quad (B_1)$$

- $L_0 < 0 < L$ and the nonlinearity f satisfies the following properties

$$f \in C[\phi(L_0), \infty), \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0. \quad (B_2)$$

- Moreover, f oscillates in the following way:

$$xf(x) > 0 \text{ for } x \in (\phi(L_0), \phi(L)) \setminus \{0\}, \quad f(x) \leq 0 \text{ for } x > \phi(L). \quad (B_3)$$

- Finally, $p \in C[0, \infty) \cap C^1(0, \infty)$ is an increasing function, that is,

$$p(0) = 0 \quad \text{and} \quad p'(t) > 0 \text{ for } t \in (0, \infty). \quad (B_4)$$

Moreover, in order to derive the main existence results about unbounded solutions of problem (9.1.1)–(9.1.2), we introduce the following auxiliary equation with a bounded nonlinearity

$$(p(t) \phi(u'(t)))' + p(t) \tilde{f}(\phi(u(t))) = 0, \quad t \in (0, \infty), \quad (9.2.1)$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \quad x > \phi(L). \end{cases} \quad (9.2.2)$$

Since \tilde{f} is bounded on \mathbb{R} , the maximal interval of existence for each solution of problem (9.2.1), (9.1.2) is $[0, \infty)$.

In this section, we collect preliminary results for solutions of problem (9.2.1), (9.1.2) derived in [13]. Properties, asymptotic behaviour and a priori estimates of such solutions are specified in Lemmas 9.2.1–9.2.9. The existence and continuous dependence on initial values of solutions are provided in Theorems 9.2.10 and 9.2.12, respectively.

Lemma 9.2.1 ([13, Lemma 2.1 b]). *Let (B_1) – (B_4) hold and let u be a solution of equation (9.2.1). Assume that there exists $a \geq 0$ such that $u(a) \in (L_0, 0)$ and $u'(a) = 0$. Then $u'(t) > 0$ for $t \in (a, \theta]$, where θ is the first zero of u on (a, ∞) . If such θ does not exist, then $u'(t) > 0$ for $t \in (a, \infty)$.*

Lemma 9.2.2 ([13, Lemma 2.2]). *Let (B_1) – (B_4) hold and let u be a solution of equation (9.2.1). Assume that there exists $a \geq 0$ such that $u(a) = L$ and $u'(a) = 0$.*

a) *Let $\theta > a$ be the first zero of u on (a, ∞) . Then there exists $a_1 \in [a, \theta)$ such that*

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 \leq u(t) < L, \quad u'(t) < 0, \quad t \in (a_1, \theta].$$

b) *Let $u > 0$ on $[a, \infty)$ and $u \not\equiv L$ on $[a, \infty)$. Then there exists $a_1 \in [a, \infty)$ such that*

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L, \quad u'(t) < 0, \quad t \in (a_1, \infty).$$

In both cases, $u(t) = L$ for $t \in [a, a_1]$.

Lemma 9.2.3 ([13, Lemma 2.6]). *Assume (B_1) – (B_4) ,*

$$\lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} = 0, \quad (9.2.3)$$

and

$$\exists \bar{B} \in (L_0, 0): \tilde{F}(\bar{B}) = \tilde{F}(L), \text{ where } \tilde{F}(x) = \int_0^x \tilde{f}(\phi(s)) \, ds, \quad x \in \mathbb{R}. \quad (9.2.4)$$

Let u be a solution of equation (9.2.1) and let there exist $b \geq 0$ and $\theta > b$ such that

$$u(b) \in [\bar{B}, 0), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0, \quad t \in [b, \theta). \quad (9.2.5)$$

Then there exists $a \in (\theta, \infty)$ such that

$$u'(a) = 0, \quad u'(t) > 0, \quad t \in (b, a), \quad u(a) \in (0, L).$$

Lemma 9.2.4 ([13, Lemma 2.7]). Assume that hypotheses (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Let u be a solution of equation (9.2.1) and let $a \geq 0$ and $\theta > a$ be such that

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0, \quad t \in [a, \theta). \quad (9.2.6)$$

Then there exists $b \in (\theta, \infty)$ such that

$$u'(b) = 0, \quad u'(t) < 0, \quad t \in (a, b), \quad u(b) \in (\bar{B}, 0).$$

Lemma 9.2.5 ([13, Lemma 2.8]). Assume that (B_1) – (B_4) and (9.2.3) hold. Let u be a solution of equation (9.2.1) and let $b \geq 0$ be such that

$$u(b) \in (L_0, 0), \quad u'(b) = 0, \quad u(t) < 0, \quad t \in [b, \infty).$$

Then

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0.$$

Lemma 9.2.6 ([13, Lemma 2.9]). Assume that (B_1) – (B_4) and (9.2.3) hold. Let u be a solution of equation (9.2.1) and let $a \geq 0$ be such that

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(t) > 0, \quad t \in [a, \infty).$$

Then either

$$u(t) = L, \quad t \in [a, \infty)$$

or

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0.$$

Lemma 9.2.7 ([13, Lemma 3.1]). Assume that hypotheses (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Let u be a solution of problem (9.2.1), (9.1.2) with $u_0 \in (L_0, \bar{B})$. Let there exist $\theta > 0$, $a > \theta$ such that

$$u(\theta) = 0, \quad u(t) < 0, \quad t \in [0, \theta)$$

and

$$u'(a) = 0, \quad u'(t) > 0, \quad t \in (\theta, a).$$

Then

$$u(a) \in (0, L] \quad \text{and} \quad u'(t) > 0, \quad t \in (0, a).$$

Lemma 9.2.8 ([13, Lemma 3.2]). *Let assumptions (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Let u be a solution of problem (9.2.1), (9.1.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then*

$$u_0 \in [\bar{B}, 0) \cup (0, L) \text{ implies that } \bar{B} < u(t) < L, \quad t \in (0, \infty)$$

and

$$u_0 \in (L_0, \bar{B}) \text{ implies that } u_0 < u(t), \quad t \in (0, \infty).$$

For the following result, we introduce a function φ defined as

$$\varphi(t) := \begin{cases} \frac{1}{p(t)} \int_0^t p(s) \, ds, & t \in (0, T] \\ 0, & t = 0. \end{cases} \quad (9.2.7)$$

This function is continuous on $[0, T]$ and satisfies that

$$0 < \varphi(t) \leq t, \quad t \in (0, T] \quad \text{and} \quad \lim_{t \rightarrow 0^+} \varphi(t) = 0. \quad (9.2.8)$$

Moreover, since \tilde{f} is bounded, there exists some constant $\widetilde{M} > 0$ such that

$$|\tilde{f}(x)| \leq \widetilde{M}, \quad \forall x \in \mathbb{R}. \quad (9.2.9)$$

Lemma 9.2.9 ([13, Lemma 3.4]). *Assume (B_1) – (B_4) . Let u be a solution of problem (9.2.1), (9.1.2) with $u_0 \in [L_0, L]$. The inequality*

$$\int_0^\beta \frac{p'(t)}{p(t)} |\phi(u'(t))| \, dt \leq \widetilde{M} (\beta - \varphi(\beta)),$$

with φ given in (9.2.7), is valid for every $\beta > 0$. If moreover (9.2.3) and (9.2.4) hold, then there exists $\widetilde{c} > 0$ such that

$$|u'(t)| \leq \widetilde{c}, \quad t \in [0, \infty),$$

for every solution u of (9.2.1), (9.1.2) with $u_0 \in (L_0, 0) \cup (0, L)$.

The existence of solutions of the auxiliary problem (9.2.1), (9.1.2) is proved in [13] by means of the Schauder's fixed point Theorem. We state this existence result in the next theorem.

Theorem 9.2.10 ([13, Theorem 4.1]). *Assume that conditions (B_1) – (B_4) hold. Then, for each $u_0 \in [L_0, L]$, there exists a solution u of problem (9.2.1), (9.1.2).*

Remark 9.2.11. *Assumption (B_2) yields that the constant functions $u \equiv L_0$, $u \equiv 0$ and $u \equiv L$ are solutions of problem (9.1.1)–(9.1.2) on $[0, \infty)$ with $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$, respectively.*

The uniqueness of solutions of (9.2.1), (9.1.2) follows from the continuous dependence on initial values. This assertion is based on the Lipschitz property of both f and ϕ .

Theorem 9.2.12 ([13, Theorem 4.3]). *Assume (B_1) – (B_4) and*

$$f \in \text{Lip} [\phi(L_0), \phi(L)], \quad (9.2.10)$$

$$\phi^{-1} \in \text{Lip}_{\text{loc}}(\mathbb{R}). \quad (9.2.11)$$

Let u_i be a solution of problem (9.2.1), (9.1.2) with $u_0 = B_i \in [L_0, L]$, $i = 1, 2$. Then, for each $\beta > 0$, there exists $K > 0$ such that

$$\|u_1 - u_2\|_{C^1[0, \beta]} \leq K |B_1 - B_2|.$$

Furthermore, any solution of problem (9.2.1), (9.1.2), with $u_0 \in [L_0, L]$, is unique.

Remark 9.2.13. *We note that, even if (9.2.10) and (9.2.11) do not hold, $u \equiv 0$ is the unique solution of (9.1.1)–(9.1.2) with $u_0 = 0$. Indeed, if $u(0) = 0$, then u' cannot be positive on $(0, \delta)$ for any $\delta > 0$ since, in such a case, u would be positive on $(0, \delta)$ and integrating equation (9.1.1) from 0 to $t \in (0, \delta)$, we would get, by (B_3) , that*

$$p(t) \phi(u'(t)) = - \int_0^t p(s) f(\phi(u(s))) \, ds < 0,$$

which is a contradiction. Similarly, u' cannot be negative. Therefore, the solution $u(t) \equiv 0$ is the unique solution of problem (9.1.1)–(9.1.2) with $u_0 = 0$ and, clearly, it is a damped solution.

Remark 9.2.14. *We note that Lemmas 9.2.3–9.2.8 are proved in [13] under the weaker assumption*

$$\limsup_{t \rightarrow \infty} \frac{p'(t)}{p(t)} < \infty \quad (9.2.12)$$

instead of condition (9.2.3). Obviously (9.2.3) implies (9.2.12) but, since in this chapter we will need to assume the stronger condition (9.2.3), we have decided to use it in the formulations of all the results in this section, for the sake of keeping the formulations as simple as possible.

Similarly, no sign conditions on $f(x)$, $x \notin [L_0, L]$ are needed in [13] while here we use (B_3) . In particular, contrary to [13], we will need the condition $f(x) \leq 0$ for $x > \phi(L)$. So, we will use this additional condition in formulations of results in this section, whereas these results are proved in [13] without it.

9.3. Auxiliary Results

In this section, we provide auxiliary lemmas, which will be used in Section 9.4 for proving the existence of escape solutions of the auxiliary problem (9.2.1), (9.1.2).

Note that all the solutions of problem (9.2.1), (9.1.2) with $u_0 \in [\bar{B}, L)$ are damped solutions (see Remark 9.2.13 and Lemma 9.2.8). Therefore, when looking for escape solutions of problem (9.2.1), (9.1.2), we will consider only $u_0 \in [L_0, \bar{B})$. Such solutions can be equivalently characterized as follows.

Lemma 9.3.1. *Let (B_1) – (B_4) , (9.2.3) and (9.2.4) hold and let u be a solution of problem (9.2.1), (9.1.2). Then u is an escape solution if and only if*

$$\sup \{u(t) : t \in [0, \infty)\} > L. \quad (9.3.1)$$

Proof. Let u fulfil (9.3.1). According to Definition 9.1.2, u is not a damped solution and hence, due to Lemma 9.2.8, $u(0) < \bar{B} < 0$. Consequently, there exists a maximal $c > 0$ such that $u(t) < L$ for $t \in [0, c)$ and

$$u(c) = L, \quad u'(c) \geq 0.$$

Assume that $u'(c) = 0$. Using Lemma 9.2.2 (and, in case of more roots of u , also Lemmas 9.2.3 and 9.2.4), we get that

$$\sup \{u(t) : t \in [0, \infty)\} = u(c) = L,$$

contrary to (9.3.1). Therefore, $u'(c) > 0$ and so u is an escape solution.

On the other hand, if u is an escape solution of problem (9.2.1), (9.1.2), then (9.3.1) follows immediately from Definition 9.1.3. \square

The proof of the existence of escape solutions are based on Lemmas 9.3.2 and 9.3.5. These lemmas are denoted here as basic lemmas because they are essential for the proof of existence of escape solutions.

Basic Lemma I (Lemma 9.3.2), fully covers the case when the uniqueness of solutions of (9.2.1), (9.1.2) is guaranteed. In particular, in such a case, $u \equiv L_0$ is the unique solution with $u_0 = L_0$. Therefore, $u_0 = L_0$ is not discussed in the context of escape solutions.

The situation is different when (9.2.10) and (9.2.11) do not hold, as we will see in Basic Lemma II (Lemma 9.3.5).

Lemma 9.3.2 (Basic Lemma I). *Let (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Choose $C \in (L_0, \bar{B})$ and a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. For each $n \in \mathbb{N}$, let u_n be a*

solution of problem (9.2.1), (9.1.2) with $u_0 = B_n$ and let $(0, b_n)$ be the maximal interval such that

$$u_n(t) < L \quad \text{and} \quad u'_n(t) > 0, \quad t \in (0, b_n). \quad (9.3.2)$$

Finally, let $\gamma_n \in (0, b_n)$ be such that

$$u_n(\gamma_n) = C. \quad (9.3.3)$$

If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (9.2.1), (9.1.2).

Proof. If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then there exists a subsequence which goes to infinity as $n \rightarrow \infty$. For simplicity, let us denote it also by $\{\gamma_n\}_{n=1}^{\infty}$. This way, we have that

$$\lim_{n \rightarrow \infty} \gamma_n = \infty, \quad \gamma_n < b_n, \quad n \in \mathbb{N}.$$

Assume now, on the contrary, that u_n is not an escape solution of problem (9.2.1), (9.1.2) for any $n \in \mathbb{N}$. Then, by Lemma 9.3.1,

$$\sup \{u_n(t) : t \in [0, \infty)\} \leq L, \quad \forall n \in \mathbb{N}. \quad (9.3.4)$$

We will divide the proof into several steps:

Step 1: It holds that $u_n(b_n) \in [0, L]$ and $u'_n(b_n) = 0$ for all $n \in \mathbb{N}$.

Fixed $n \in \mathbb{N}$ and consider a solution u_n of problem (9.2.1), (9.1.2) with $u_0 = B_n$.

First assume that $u_n < 0$ on $[0, \infty)$. Then, by Lemma 9.2.1, we get that $u'_n > 0$ on $(0, \infty)$. This way, the interval $(0, b_n)$ given in (9.3.2) is $(0, \infty)$. In addition, from Lemma 9.2.5, we get that

$$\lim_{t \rightarrow \infty} u_n(t) = 0, \quad \lim_{t \rightarrow \infty} u'_n(t) = 0.$$

If we put

$$\lim_{t \rightarrow \infty} u_n(t) =: u_n(b_n) \quad \text{and} \quad \lim_{t \rightarrow \infty} u'_n(t) =: u'_n(b_n),$$

we obtain

$$u_n(b_n) = 0, \quad u'_n(b_n) = 0. \quad (9.3.5)$$

Now assume that u_n changes sign and let $\theta > 0$ be the first zero of u_n . By Lemma 9.2.1, $u'_n > 0$ on $(0, \theta]$. We will consider two cases:

- (i) Let $u'_n > 0$ on (θ, ∞) . Since, according to (9.3.4), $0 < u_n < L$ on (θ, ∞) , then $(0, b_n) = (0, \infty)$. We will prove that, in these conditions,

$$\lim_{t \rightarrow \infty} u_n(t) = L \quad \text{and} \quad \lim_{t \rightarrow \infty} u'_n(t) = 0.$$

We denote

$$\lim_{t \rightarrow \infty} u_n(t) =: \ell \in (0, L].$$

Since u_n is a solution of equation (9.2.1), then

$$\phi'(u'_n(t)) u''_n(t) + \frac{p'(t)}{p(t)} \phi(u'_n(t)) + \tilde{f}(\phi(u_n(t))) = 0, \quad t \in (0, \infty). \quad (9.3.6)$$

If we restrict the previous equation to the interval (θ, ∞) then, by (B_1) – (B_4) , we have that

$$\frac{p'(t)}{p(t)} \phi(u'_n(t)) > 0, \quad \tilde{f}(\phi(u_n(t))) > 0 \quad \text{and} \quad \phi'(u'_n(t)) > 0,$$

so we deduce that

$$u''_n(t) < 0, \quad t \in (\theta, \infty).$$

Consequently, u'_n is decreasing on (θ, ∞) and so, there must exist

$$\lim_{t \rightarrow \infty} u'_n(t) \geq 0.$$

If $\lim_{t \rightarrow \infty} u'_n(t) = a > 0$, then $\lim_{t \rightarrow \infty} u_n(t) = \infty$, which is a contradiction. Therefore,

$$\lim_{t \rightarrow \infty} u'_n(t) = 0.$$

Now, assume that $\ell \in (0, L)$. Letting $t \rightarrow \infty$ in (9.3.6), we get, by (B_1) and (9.2.3), that

$$\phi'(0) \lim_{t \rightarrow \infty} u''_n(t) = -\tilde{f}(\phi(\ell)).$$

From (B_3) , $\ell \in (0, L)$ implies that $\tilde{f}(\phi(\ell)) \in (0, \infty)$ and we get

$$\lim_{t \rightarrow \infty} u''_n(t) < 0,$$

which contradicts that $\lim_{t \rightarrow \infty} u'_n(t) = 0$. Therefore, necessarily, $\ell = L$.

Thus, we conclude that

$$u_n(b_n) = L \quad \text{and} \quad u'_n(b_n) = 0. \quad (9.3.7)$$

- (ii) Let $a > \theta$ be the first zero of u'_n . By (9.3.4) we have $u_n(a) \leq L$. For $b_n = a$ we get (9.3.2) and

$$u_n(b_n) \in (0, L], \quad u'_n(b_n) = 0. \quad (9.3.8)$$

Step 2: We will prove that

$$\frac{\tilde{F}(C) - \tilde{F}(L)}{L - C} \frac{1}{K_n} < \phi(u'_n(\bar{\gamma}_n)), \quad (9.3.9)$$

where

$$K_n := \sup \left\{ \frac{p'(t)}{p(t)} : t \in [\gamma_n, b_n] \right\}$$

and $\bar{\gamma}_n \in [\gamma_n, b_n]$ is such that

$$u'_n(\bar{\gamma}_n) = \max \{ u'_n(t) : t \in [\gamma_n, b_n] \}. \quad (9.3.10)$$

Note that, due to (9.2.3), $\lim_{n \rightarrow \infty} K_n = 0$.

Let n be fixed. We define

$$E_n(t) := \int_0^{u'_n(t)} x \phi'(x) dx + \tilde{F}(u_n(t)), \quad t \in (0, b_n).$$

Then, by (9.3.6), the following inequality holds for $t \in (0, b_n)$,

$$\begin{aligned} \frac{d E_n(t)}{d t} &= u'_n(t) \phi'(u'_n(t)) u''_n(t) + \tilde{f}(\phi(u_n(t))) u'_n(t) \\ &= -\frac{p'(t)}{p(t)} \phi(u'_n(t)) u'_n(t) < 0, \end{aligned}$$

where the negative sign is deduced from (B_1) , (B_4) and (9.3.2).

Integrating the above equality over (γ_n, b_n) and using (9.3.2) and (9.3.10), we obtain

$$\begin{aligned} E_n(\gamma_n) - E_n(b_n) &= \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} \phi(u'_n(t)) u'_n(t) dt \leq \phi(u'_n(\bar{\gamma}_n)) \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} u'_n(t) dt \\ &\leq \phi(u'_n(\bar{\gamma}_n)) K_n \int_{\gamma_n}^{b_n} u'_n(t) dt \leq \phi(u'_n(\bar{\gamma}_n)) K_n (L - C). \end{aligned}$$

Hence,

$$E_n(\gamma_n) \leq E_n(b_n) + \phi(u'_n(\bar{\gamma}_n)) K_n (L - C).$$

Moreover, from Step 1, we deduce

$$E_n(\gamma_n) > \tilde{F}(u_n(\gamma_n)) = \tilde{F}(C) \quad \text{and} \quad E_n(b_n) = \tilde{F}(u_n(b_n)) \leq \tilde{F}(L).$$

This leads to

$$\tilde{F}(C) < E_n(\gamma_n) \leq \tilde{F}(L) + \phi(u'_n(\bar{\gamma}_n)) K_n (L - C). \quad (9.3.11)$$

Hence, we derive the estimate (9.3.9).

Step 3: Existence of an escape solution of problem (9.2.1), (9.1.2).

Consider a sequence $\{u_n\}_{n=1}^\infty$. Since $\lim_{n \rightarrow \infty} K_n = 0$, we derive from (9.3.9) that

$$\lim_{n \rightarrow \infty} \phi(u'_n(\bar{\gamma}_n)) = \infty. \quad (9.3.12)$$

Using (B_1) , we obtain that

$$\lim_{n \rightarrow \infty} u'_n(\bar{\gamma}_n) = \lim_{n \rightarrow \infty} \phi^{-1}(\phi(u'_n(\bar{\gamma}_n))) = \infty.$$

Since $\tilde{F} \geq 0$ (with \tilde{F} given in (9.2.4)) and E_n is decreasing on $(0, b_n)$, using (9.3.11), we obtain the following inequality for all $n \in \mathbb{N}$

$$\int_0^{u'_n(\bar{\gamma}_n)} x \phi'(x) dx \leq E_n(\bar{\gamma}_n) \leq E_n(\gamma_n) \leq \tilde{F}(L) + \phi(u'_n(\bar{\gamma}_n)) K_n (L - C)$$

and, therefore,

$$\lim_{n \rightarrow \infty} \left(\int_0^{u'_n(\bar{\gamma}_n)} x \phi'(x) dx - \phi(u'_n(\bar{\gamma}_n)) K_n (L - C) \right) \leq \tilde{F}(L) < \infty.$$

Since

$$\lim_{n \rightarrow \infty} u'_n(\bar{\gamma}_n) = \infty,$$

then there exists $n_0 \in \mathbb{N}$ such that

$$u'_n(\bar{\gamma}_n) > 1, \quad n \geq n_0.$$

Therefore, for $n \geq n_0$,

$$\int_0^{u'_n(\bar{\gamma}_n)} x \phi'(x) dx > \int_1^{u'_n(\bar{\gamma}_n)} x \phi'(x) dx > \int_1^{u'_n(\bar{\gamma}_n)} \phi'(x) dx = \phi(u'_n(\bar{\gamma}_n)) - \phi(1).$$

By (9.3.12) and $\lim_{n \rightarrow \infty} K_n = 0$ we derive

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\int_0^{u'_n(\bar{\gamma}_n)} x \phi'(x) dx - \phi(u'_n(\bar{\gamma}_n)) K_n (L - C) \right) \\ & \geq \lim_{n \rightarrow \infty} \phi(u'_n(\bar{\gamma}_n)) (1 - K_n (L - C)) - \phi(1) = \infty. \end{aligned}$$

This yields a contradiction. Therefore, the sequence $\{u_n\}_{n=1}^\infty$ contains an escape solution of problem (9.2.1), (9.1.2). \square

The previous lemma gives a powerful tool to ensure the existence of escape solutions with $u_0 \in (L_0, C)$.

However, if ϕ^{-1} and f are not Lipschitz continuous, then problem (9.2.1), (9.1.2) with $u_0 \in [L_0, L] \setminus \{0\}$ can have multiple solutions. These solutions may be escape solutions. In particular, more solutions can start at L_0 , not only the constant solution $u \equiv L_0$. Therefore, we need to extend the assertions of Lemma 9.3.2, which deal with values greater than L_0 , to the case $u_0 = L_0$. For this purpose next two lemmas will be helpful.

Lemma 9.3.3. *Let (B_1) – (B_4) hold and let u be a solution of problem (9.2.1), (9.1.2) such that*

$$u_0 = L_0, \quad u \not\equiv L_0, \quad u(t) \geq L_0 \text{ for } t \in [0, \infty). \quad (9.3.13)$$

Then there exists $a \geq 0$ such that

$$u(t) = L_0 \text{ for } t \in [0, a] \quad (9.3.14)$$

and

$$u'(t) > 0 \text{ for } t \in (a, \theta],$$

where θ is the first zero of u in (a, ∞) . If such θ does not exist, then

$$u'(t) > 0 \text{ for } t \in (a, \infty).$$

If $\theta \in (a, \infty)$ and there exist $a_1 > \theta$ such that

$$u'(a_1) = 0 \quad \text{and} \quad u'(t) > 0, \quad t \in (\theta, a_1), \quad (9.3.15)$$

then $u(a_1) \in (0, L]$.

Proof. By (9.3.13), there exists $\tau > 0$ such that

$$L_0 < u(\tau) < 0.$$

Put

$$a := \inf \{ \tau > 0 : L_0 < u(\tau) < 0 \}.$$

Then u fulfils (9.3.14) and $u'(a) = 0$.

Put

$$\theta := \sup \{ \tau > a : L_0 < u(\tau) < 0 \}.$$

Then

$$p(t) \tilde{f}(\phi(u(t))) < 0, \quad t \in (a, \theta). \quad (9.3.16)$$

Integrating equation (9.2.1) over $[a, t]$, we get, by (9.3.16),

$$p(t) \phi(u'(t)) = - \int_a^t p(s) \tilde{f}(\phi(u(s))) \, ds > 0, \quad t \in (a, \theta) \quad (9.3.17)$$

and, since $p(t) > 0$, necessarily $u'(t) > 0$ for $t \in (a, \theta)$.

If $\theta = \infty$, then the proof is finished.

On the other hand, if $\theta < \infty$, then θ is the first zero of u on (a, ∞) and (9.3.17) yields $u'(\theta) > 0$.

Let $\theta \in (a, \infty)$ and $a_1 > \theta$ such that (9.3.15) holds. Since $u(\theta) = 0$ and $u'(t) > 0$ on (θ, a_1) , then $u(a_1) > 0$. Assume that $u(a_1) > L$. Then there exists $a_0 \in (\theta, a_1)$ such that $u > L$ on $(a_0, a_1]$. Integrating equation (9.2.1) over (a_0, a_1) and using (9.2.2), we obtain

$$p(a_0) \phi(u'(a_0)) - p(a_1) \phi(u'(a_1)) = \int_{a_0}^{a_1} p(s) \tilde{f}(\phi(u(s))) \, ds = 0,$$

and so, $p(a_0) \phi(u'(a_0)) = 0$. Consequently, $u'(a_0) = 0$, which contradicts that $u' > 0$ on (θ, a_1) . We have proved that $u(a_1) \leq L$, which completes the proof. \square

Lemma 9.3.4. *Let (B_1) – (B_4) and (9.2.3) hold and let u be a solution of (9.2.1), (9.1.2) satisfying that*

$$u_0 = L_0, \quad u \neq L_0, \quad u(t) \geq L_0 \text{ for } t \in [0, \infty).$$

Assume that

$$u(t) < 0, \quad t \in [0, \infty).$$

Then

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0.$$

Proof. By Lemma 9.3.3, there exists $a \geq 0$ such that $u(t) = L_0$ for $t \in [0, a]$ and $u'(t) > 0$ for $t \in (a, \infty)$. Hence, u is increasing on (a, ∞) and so

$$L_0 < u(t) < 0, \quad t \in (a, \infty)$$

and there exists

$$\lim_{t \rightarrow \infty} u(t) =: \ell \in (L_0, 0].$$

Now, if u is a solution of (9.2.1), then

$$\phi'(u'(t)) u''(t) + \frac{p'(t)}{p(t)} \phi(u'(t)) + \tilde{f}(\phi(u(t))) = 0, \quad t \in (0, \infty). \quad (9.3.18)$$

Multiplying previous equation by u' and integrating it from a to t , we obtain

$$\psi_1(t) + \psi_2(t) + \psi_3(t) = 0, \quad t \in (a, \infty),$$

where

$$\begin{aligned} \psi_1(t) &= \int_{u'(a)}^{u'(t)} x \phi'(x) \, dx, \\ \psi_2(t) &= \int_a^t \frac{p'(s)}{p(s)} \phi(u'(s)) u'(s) \, ds, \\ \psi_3(t) &= \int_{u(a)}^{u(t)} \tilde{f}(\phi(x)) \, dx. \end{aligned}$$

It holds that

$$\psi_3(t) = \tilde{F}(u(t)) - \tilde{F}(u(a)).$$

Since $\tilde{F}(x)$ is decreasing for $x \in (L_0, 0)$ and u is increasing on (a, ∞) , $\tilde{F}(u(t))$ is decreasing for $t \in (a, \infty)$ and so

$$\lim_{t \rightarrow \infty} \tilde{F}(u(t)) = \tilde{F}(\ell).$$

Therefore,

$$\lim_{t \rightarrow \infty} \psi_3(t) =: Q_3 \in (-\tilde{F}(L_0), 0).$$

On the other hand, since ϕ_1 is positive on (a, ∞) , it occurs that $\psi_2(t) < \psi_3(t)$ for $t \in (a, \infty)$. This way, since ψ_2 is continuous, increasing and positive on (a, ∞) , it holds that

$$\lim_{t \rightarrow \infty} \psi_2(t) =: Q_2 \in (0, -Q_3].$$

As a consequence, we get that

$$\lim_{t \rightarrow \infty} \psi_1(t) =: Q_1 \in [0, -\tilde{F}(L_0)).$$

Thus, defining

$$\Phi(z) := \int_0^z x \phi'(x) \, dx,$$

it occurs that $\lim_{t \rightarrow \infty} \Phi(u'(t)) = Q_1$. Moreover, since Φ is positive, continuous and increasing on $(0, \infty)$, its inverse Φ^{-1} is also positive, continuous and increasing. Consequently,

$$\lim_{t \rightarrow \infty} u'(t) = \lim_{t \rightarrow \infty} \Phi^{-1}(\Phi(u'(t))) = \Phi^{-1}(Q_1) \geq 0$$

and, since there exists $\lim_{t \rightarrow \infty} u(t) =: \ell \in (L_0, 0]$, we conclude that

$$\lim_{t \rightarrow \infty} u'(t) = 0.$$

Now, assume that $\ell \neq 0$. Then, taking the limit when t goes to ∞ in (9.3.18) and using (9.2.3), we obtain

$$\phi'(0) \lim_{t \rightarrow \infty} u''(t) = -\tilde{f}(\phi(l)).$$

Since $-\tilde{f}(\phi(l)) \in (-\infty, 0)$, then necessarily $\lim_{t \rightarrow \infty} u''(t) > 0$, which is a contradiction with $\lim_{t \rightarrow \infty} u'(t) = 0$. Therefore, $\ell = 0$ and the result is proved. \square

Lemma 9.3.5 (Basic Lemma II). *Let $(B_1)-(B_4)$, (9.2.3) and (9.2.4) hold. Choose $C \in (L_0, \bar{B})$. For each $n \in \mathbb{N}$, let u_n be a solution of problem (9.2.1), (9.1.2) with $u_0 = L_0$ and let (a_n, b_n) be the maximal interval such that*

$$L_0 < u_n(t) < L \quad \text{and} \quad u'_n(t) > 0, \quad t \in (a_n, b_n).$$

Finally, let $\gamma_n \in (a_n, b_n)$ be such that

$$u_n(\gamma_n) = C.$$

If the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded, then the sequence $\{u_n\}_{n=1}^\infty$ contains an escape solution of problem (9.2.1), (9.1.2) with $u_0 = L_0$.

Proof. The proof is held in an analogous way to the proof of Lemma 9.3.2 where in Step 1, Lemmas 9.3.3 and 9.3.4 are used instead of Lemmas 9.2.1 and 9.2.5, respectively. \square

9.4. Existence of Escape Solutions

This section is devoted to prove the existence of escape solutions of problem (9.2.1), (9.1.2).

First, we will discuss the existence of escape solutions provided the Lipschitz continuity of ϕ^{-1} and f . For this purpose we choose a sequence of solutions which converges locally uniformly to the constant solution $u \equiv L_0$. In this manner we obtain an unbounded sequence $\{\gamma_n\}_{n=1}^\infty$ required in the Basic Lemma I (Lemma 9.3.2) for the existence of an escape solution.

This approach fails without the assumption on the Lipschitz condition. This situation is subject of investigation in the rest of this section. In particular, we will solve this problem with the lower and upper solutions method.

Theorem 9.4.1 (Existence of escape solutions of problem (9.2.1), (9.1.2) I). *Assume that $(B_1)-(B_4)$, (9.2.3), (9.2.4), (9.2.10) and (9.2.11) hold. Then there exist infinitely many escape solutions of problem (9.2.1), (9.1.2) with different starting values in (L_0, \bar{B}) .*

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorems 9.2.10 and 9.2.12, there exists a unique solution u_n of problem (9.2.1), (9.1.2) with $u_0 = B_n$.

By Lemma 9.2.1, there exists a maximal $a_n > 0$ such that $u'_n > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then

$$u_n(t) < 0, \quad u'_n(t) > 0, \quad t \in (0, b_n).$$

Further, due to Lemmas 9.2.1 and 9.2.5, either $\lim_{t \rightarrow \infty} u_n(t) = 0$ or u_n has a zero $\theta_n \in (0, b_n)$. Consequently, there exists $\gamma_n \in (0, b_n)$ satisfying that $u_n(\gamma_n) = C$.

This way, from the sequence $\{B_n\}_{n=1}^\infty \subset (L_0, C)$, we get the sequence $\{u_n\}_{n=1}^\infty$ of solutions of problem (9.2.1), (9.1.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^\infty$.

Assume that $\lim_{n \rightarrow \infty} B_n = L_0$. Then, by Theorem 9.2.12, the sequence $\{u_n\}_{n=1}^\infty$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n \rightarrow \infty} \gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded.

Thus, by Lemma 9.3.2 there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (9.2.1), (9.1.2). We have $u_{n_0}(0) = B_{n_0} > L_0$.

Now, consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^\infty$. By Lemma 9.3.2 there exists $n_1 \in \mathbb{N}$ such that u_{n_1} is an escape solution of problem (9.2.1), (9.1.2) such that $u_{n_1}(0) = B_{n_1} > L_0$.

Repeating this procedure, we obtain the sequence $\{u_{n_k}\}_{k=0}^\infty$ of escape solutions of problem (9.2.1), (9.1.2). \square

Remark 9.4.2. *We note that the proof of previous theorem does not remain valid if we eliminate hypotheses (9.2.10) and (9.2.11). The reason is that, without these hypotheses, we do not have uniqueness of solution. Then, in the previous proof, $\lim_{n \rightarrow \infty} B_n = L_0$ implies that the sequence $\{u_n\}_{n=1}^\infty$ converges locally uniformly on $[0, \infty)$ to a function u such that $u(0) = L_0$. However, since there is no uniqueness of solution, we can not affirm that $u \equiv L_0$ and so we can not ensure that the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded.*

Since previous method is not valid in case ϕ^{-1} and f are not Lipschitz continuous, we need to find an alternative approach to investigate the existence of escape solutions in such a case. In order to prove this existence result, we consider the lower

and upper solutions method for an auxiliary mixed problem on $[0, T]$. In particular, we will use this method to find solutions of (9.2.1) which satisfy that

$$u'(0) = 0, \quad u(T) = C, \quad C \in [L_0, L]. \quad (9.4.1)$$

Definition 9.4.3. A function $u \in C^1([0, T])$ with $\phi(u') \in C^1((0, T])$ is a solution of problem (9.2.1), (9.4.1) if u fulfills (9.2.1) for $t \in (0, T]$ and satisfies (9.4.1).

Definition 9.4.4. A function $\sigma_1 \in C([0, T])$ is a lower solution of problem (9.2.1), (9.4.1) if there exists a finite (possibly empty) set $\Sigma_1 \subset (0, T)$ such that $\sigma_1 \in C^2((0, T] \setminus \Sigma_1)$ and

$$(p(t) \phi(\sigma_1'(t)))' + p(t) \tilde{f}(\phi(\sigma_1(t))) \geq 0, \quad t \in (0, T] \setminus \Sigma_1, \quad (9.4.2)$$

$$-\infty < \sigma_1'(\tau^-) < \sigma_1'(\tau^+) < \infty, \quad \tau \in \Sigma_1, \quad (9.4.3)$$

$$\sigma_1'(0^+) \geq 0, \quad \sigma_1(T) \leq C. \quad (9.4.4)$$

Analogously,

Definition 9.4.5. A function $\sigma_2 \in C([0, T])$ is an upper solution of problem (9.2.1), (9.4.1) if there exists a finite (possibly empty) set $\Sigma_2 \subset (0, T)$ such that $\sigma_2 \in C^2((0, T] \setminus \Sigma_2)$ and

$$(p(t) \phi(\sigma_2'(t)))' + p(t) \tilde{f}(\phi(\sigma_2(t))) \leq 0, \quad t \in (0, T] \setminus \Sigma_2, \quad (9.4.5)$$

$$-\infty < \sigma_2'(\tau^+) < \sigma_2'(\tau^-) < \infty, \quad \tau \in \Sigma_2, \quad (9.4.6)$$

$$\sigma_2'(0^+) \leq 0, \quad \sigma_2(T) \geq C. \quad (9.4.7)$$

Theorem 9.4.6 (Lower and upper solutions method). Let (B_1) – (B_4) hold and let σ_1 and σ_2 be lower and upper solutions of problem (9.2.1), (9.4.1) such that

$$\sigma_1(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Then problem (9.2.1), (9.4.1) has a solution u such that

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Proof. The proof is divided into two steps.

Step 1: Construction of an auxiliary problem and its solvability.

For $t \in [0, T]$ and $x \in \mathbb{R}$ we define the following auxiliary nonlinearity

$$f^*(t, x) = \begin{cases} \tilde{f}(\phi(\sigma_1(t))) + \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, & x < \sigma_1(t), \\ \tilde{f}(\phi(x)), & \sigma_1(t) \leq x \leq \sigma_2(t), \\ \tilde{f}(\phi(\sigma_2(t))) - \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}, & x > \sigma_2(t). \end{cases}$$

Note that f^* is bounded, that is, there exists $M^* > 0$ such that

$$|f^*(t, x)| \leq M^*, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \quad (9.4.8)$$

Consider the auxiliary equation

$$(p(t) \phi(u'(t)))' + p(t) f^*(t, u(t)) = 0, \quad t \in (0, T]. \quad (9.4.9)$$

Integrating (9.4.9), we get the equivalent form of problem (9.4.9), (9.4.1):

$$u(t) = C - \int_t^T \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau)) \, d\tau \right) \, ds, \quad t \in [0, T].$$

Now, consider the Banach space $\mathcal{C}([0, T])$ with the maximum norm and define an operator $\mathcal{F}: \mathcal{C}([0, T]) \rightarrow \mathcal{C}([0, T])$ in the following way:

$$(\mathcal{F}u)(t) := C - \int_t^T \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau)) \, d\tau \right) \, ds.$$

Put $\Lambda := \max\{|L_0|, L\}$ and consider the ball

$$\mathcal{B}(0, R) = \{u \in \mathcal{C}([0, T]): \|u\|_{\mathcal{C}([0, T])} \leq R\},$$

where $R := \Lambda + T \phi^{-1}(M^* T)$ and M^* is the upper bound given in (9.4.8). Since ϕ is increasing on \mathbb{R} , ϕ^{-1} is also increasing on \mathbb{R} and, by (9.2.8),

$$\phi^{-1}(M^* \varphi(t)) \leq \phi^{-1}(M^* T), \quad t \in [0, T],$$

where φ is defined in (9.2.7). Then, the norm of $\mathcal{F}u$ can be estimated as follows

$$\begin{aligned} \|\mathcal{F}u\|_{\mathcal{C}([0, T])} &= \max_{t \in [0, T]} \left| C - \int_t^T \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau)) \, d\tau \right) \, ds \right| \\ &\leq \Lambda + \int_t^T |\phi^{-1}(M^* \varphi(s))| \, ds \leq \Lambda + \int_t^T \phi^{-1}(M^* T) \, ds \\ &\leq \Lambda + T \phi^{-1}(M^* T) = R, \end{aligned}$$

which yields that \mathcal{F} maps $\mathcal{B}(0, R)$ to itself.

Let us prove that \mathcal{F} is compact on $\mathcal{B}(0, R)$.

First, we will show that \mathcal{F} is continuous. Choose a sequence $\{u_n\} \subset \mathcal{C}([0, T])$ such that $\lim_{n \rightarrow \infty} \|u_n - u\|_{\mathcal{C}([0, T])} = 0$. We have that

$$\begin{aligned} (\mathcal{F}u_n)(t) - (\mathcal{F}u)(t) &= - \int_t^T \left(\phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u_n(\tau)) \, d\tau \right) \right. \\ &\quad \left. + \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau)) \, d\tau \right) \right) \, ds. \end{aligned}$$

Since f^* is continuous on $[0, T] \times \mathbb{R}$, we get

$$\lim_{n \rightarrow \infty} \|f^*(\cdot, u_n(\cdot)) - f^*(\cdot, u(\cdot))\|_{\mathcal{C}([0, T])} = 0.$$

Now, for $n \in \mathbb{N}$, define

$$A_n(t) := \begin{cases} -\frac{1}{p(t)} \int_0^t p(\tau) f^*(\tau, u_n(\tau)) \, d\tau, & t \in (0, T], \\ 0, & t = 0 \end{cases}$$

and

$$A(t) := \begin{cases} -\frac{1}{p(t)} \int_0^t p(\tau) f^*(\tau, u(\tau)) \, d\tau, & t \in (0, T], \\ 0, & t = 0. \end{cases}$$

Then, for a fixed $n \in \mathbb{N}$,

$$|A_n(t) - A(t)| = \left| \frac{1}{p(t)} \int_0^t p(\tau) (f^*(\tau, u(\tau)) - f^*(\tau, u_n(\tau))) \, d\tau \right|, \quad t \in (0, T]$$

and, by (9.2.8) and (9.4.8), $\lim_{t \rightarrow 0^+} |A_n(t) - A(t)| = 0$. Therefore, $A_n - A \in \mathcal{C}([0, T])$ and from

$$\begin{aligned} |A_n(t) - A(t)| &\leq \|f^*(\cdot, u_n(\cdot)) - f^*(\cdot, u(\cdot))\|_{\mathcal{C}([0, T])} \left| \frac{1}{p(t)} \int_0^t p(\tau) \, d\tau \right| \\ &\leq \|f^*(\cdot, u_n(\cdot)) - f^*(\cdot, u(\cdot))\|_{\mathcal{C}([0, T])} \frac{\max \{p(\tau) : \tau \in [0, t]\}}{p(t)} t \\ &= \|f^*(\cdot, u_n(\cdot)) - f^*(\cdot, u(\cdot))\|_{\mathcal{C}([0, T])} t, \quad t \in [0, T], \end{aligned}$$

we deduce that

$$\|A_n - A\|_{\mathcal{C}([0, T])} \leq \|f^*(\cdot, u_n(\cdot)) - f^*(\cdot, u(\cdot))\|_{\mathcal{C}([0, T])} T, \quad n \in \mathbb{N}.$$

This implies that

$$\lim_{n \rightarrow \infty} \|A_n - A\|_{\mathcal{C}([0, T])} = 0.$$

Using the continuity of ϕ^{-1} on \mathbb{R} , we have

$$\lim_{n \rightarrow \infty} \|\phi^{-1}(A_n) - \phi^{-1}(A)\|_{\mathcal{C}([0, T])} = 0.$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\mathcal{F}u_n - \mathcal{F}u\|_{C([0,T])} &= \lim_{n \rightarrow \infty} \left\| \int_t^T (\phi^{-1}(A_n(s)) - \phi^{-1}(A(s))) \, ds \right\|_{C([0,T])} \\ &\leq T \lim_{n \rightarrow \infty} \|\phi^{-1}(A_n) - \phi^{-1}(A)\|_{C([0,T])} = 0, \end{aligned}$$

that is, operator \mathcal{F} is continuous.

On the other hand, choose an arbitrary $\varepsilon > 0$ and put $\delta := \frac{\varepsilon}{\phi^{-1}(M^*T)}$. Then, for $t_1, t_2 \in [0, T]$, $|t_1 - t_2| < \delta$, and $u \in \mathcal{B}(0, R)$, it holds that

$$\begin{aligned} |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)| &= \left| \int_{t_2}^{t_1} \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau)) \, d\tau \right) \, ds \right| \\ &\leq \left| \int_{t_2}^{t_1} \phi^{-1}(M^* \varphi(s)) \, ds \right| \leq \left| \int_{t_2}^{t_1} \phi^{-1}(M^*T) \, ds \right| \\ &= \phi^{-1}(M^*T) |t_1 - t_2| < \phi^{-1}(M^*T) \delta = \varepsilon. \end{aligned}$$

Hence, functions in $\mathcal{F}(\mathcal{B}(0, R))$ are equicontinuous, and, by the Arzelà–Ascoli's Theorem (Theorem 1.2.2), the set $\mathcal{F}(\mathcal{B}(0, R))$ is relatively compact. Consequently, the operator \mathcal{F} is compact on $\mathcal{B}(0, R)$.

Then, Schauder's fixed point Theorem (Theorem 1.2.3) yields the existence of a fixed point u^* of \mathcal{F} in $\mathcal{B}(0, R)$. Therefore,

$$u^*(t) = C - \int_t^T \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u^*(\tau)) \, d\tau \right) \, ds$$

is a solution of (9.4.9), (9.4.1).

Step 2: Solvability of the original problem (9.2.1), (9.4.1).

We will prove that any solution u of problem (9.4.9), (9.4.1) satisfies that

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t), \quad t \in [0, T],$$

and, therefore, it is a solution of problem (9.2.1), (9.4.1).

Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0, T]$ and assume that

$$\max\{v(t) : t \in [0, T]\} = v(t_0) > 0. \quad (9.4.10)$$

By (9.4.6), $v'(\tau^-) < v'(\tau^+)$ for each $\tau \in \Sigma_2$, so $t_0 \notin \Sigma_2$.

Moreover, $\sigma_2(T) \geq C$ and $u(T) = C$, so $v(T) \leq 0$ and, consequently, $t_0 \neq T$. Therefore, $t_0 \in [0, T] \setminus \Sigma_2$.

We distinguish two cases:

(i) If $t_0 = 0$, then (9.4.1) and (9.4.7) yield

$$v'(0^+) = u'(0^+) - \sigma'_2(0^+) = -\sigma'_2(0^+) \geq 0.$$

If $v'(0^+) > 0$, we get a contradiction with (9.4.10); hence, $v'(0^+) = 0$.

(ii) If $t_0 \in (0, T) \setminus \Sigma_2$, (9.4.10) also implies that $v'(t_0) = 0$.

Since $t_0 \in [0, T) \setminus \Sigma_2$, there exists $\delta > 0$ such that $(t_0, t_0 + \delta) \subset (0, T) \setminus \Sigma_2$ and $v(t) > 0$ for $t \in (t_0, t_0 + \delta)$. Moreover, for $t \in (t_0, t_0 + \delta)$, we have that

$$\begin{aligned} (p(t) \phi(u'(t)))' - (p(t) \phi(\sigma'_2(t)))' &\geq p(t) \left(-f^*(t, u(t)) + \tilde{f}(\phi(\sigma_2(t))) \right) \\ &= p(t) \frac{v(t)}{v(t) + 1} > 0 \end{aligned}$$

and integrating the previous expression, we obtain that

$$\int_{t_0}^t \left((p(s) \phi(u'(s)))' - (p(s) \phi(\sigma'_2(s)))' \right) ds = p(t) (\phi(u'(t)) - \phi(\sigma'_2(t))) > 0,$$

for $t \in (t_0, t_0 + \delta)$.

Therefore, since ϕ is increasing, we have that $v'(t) > 0$ on $(t_0, t_0 + \delta)$, which is a contradiction with (9.4.10).

Consequently, we have proved that

$$u(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Analogously, it can be proved that

$$u(t) \geq \sigma_1(t), \quad t \in [0, T].$$

We conclude that the solution u of problem (9.4.9), (9.4.1) is a solution of (9.2.1), (9.4.1). \square

The main result of this section (which proves the existence of escape solutions in case that ϕ^{-1} and f are not Lipschitz continuous) is contained in Theorem 9.4.8. Its proof is based on Lemmas 9.3.2 and 9.3.5, where a suitable sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (9.2.1), (9.1.2) is used. In order to get such a sequence with the starting values equal to L_0 (see part (ii) in the proof of Theorem 9.4.8), we need the next lemma.

Lemma 9.4.7. *Let (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Choose $C \in (L_0, \bar{B})$ and assume that there exists at least one solution u of problem (9.2.1), (9.1.2) satisfying that*

$$u_0 = L_0, \quad u \neq L_0, \quad u(t) \geq L_0 \text{ for } t \in [0, \infty).$$

Then there exists $\gamma > 0$ such that for each $T > \gamma$, problem (9.2.1), (9.1.2) with $u_0 = L_0$ has a solution u_T satisfying

$$u_T(T) = C, \quad u_T(t) \geq L_0, \quad t \in [0, \infty). \quad (9.4.11)$$

Proof. As a consequence of Lemmas 9.3.3 and 9.3.4, we know that either there exists $\theta > 0$ such that $u(\theta) = 0$ or $\lim_{t \rightarrow \infty} u(t) = 0$. Because of this we can take

$$\gamma = \min \{t \in [0, \infty) : u(t) = C\} > 0. \quad (9.4.12)$$

Now, fix $T > \gamma$. We will prove the assertion in four steps.

Step 1: Construction of a lower solution of problem (9.2.1), (9.4.1):

We prove that $\sigma_1 \equiv L_0$ satisfies conditions (9.4.2)–(9.4.4). First, for $t \in [0, T]$,

$$(p(t) \phi(\sigma_1'(t)))' + p(t) \tilde{f}(\phi(\sigma_1(t))) = (p(t) \phi(0))' + p(t) \tilde{f}(\phi(L_0)) = 0 \geq 0.$$

Moreover, in this case, $\sigma_1 \in \mathcal{C}^2([0, T])$, so $\Sigma_1 = \emptyset$. Finally,

$$\sigma_1'(0^+) = 0 \geq 0 \quad \text{and} \quad \sigma_1(T) = L_0 < C.$$

Therefore, σ_1 is a lower solution of (9.2.1), (9.4.1).

Step 2: Construction of an upper solution of problem (9.2.1), (9.4.1):

We distinguish two different cases.

(i) If $u < 0$ on $[0, \infty)$, we choose $\sigma_2 = u$. First,

$$(p(t) \phi(\sigma_2'(t)))' + p(t) \tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \quad t \in (0, T].$$

Moreover, in this case, $\sigma_2 \in \mathcal{C}^2((0, T])$, so $\Sigma_2 = \emptyset$. Finally,

$$\sigma_2'(0^+) = 0 \leq 0 \quad \text{and} \quad \sigma_2(T) > \sigma_2(\gamma) = C.$$

Last inequality is a consequence of the fact that, from Lemma 9.3.3, we know that σ_2 is increasing on $[a, \infty)$ for some $a \in [0, \gamma)$. Hence, σ_2 satisfies conditions (9.4.5)–(9.4.7).

(ii) If there exists $\theta > 0$ such that $u(\theta) = 0$ then $\gamma \in (0, \theta)$ and we choose

$$\sigma_2(t) = \begin{cases} u(t), & t \in [0, \theta], \\ 0, & t \in (\theta, \infty). \end{cases}$$

First,

$$(p(t) \phi(\sigma_2'(t)))' + p(t) \tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \quad t \in (0, T] \setminus \{\theta\}.$$

In this case, $\Sigma_2 = \{\theta\}$. From Lemma 9.3.3, we know that $u' > 0$ on $(a, \theta]$ for some $a \in [0, \gamma)$ and hence, $\sigma_2'(\theta^-) > 0$. It is clear that $\sigma_2'(\theta^+) = 0$, so $\sigma_2'(\theta^+) < \sigma_2'(\theta^-)$.

Finally, analogously to case (i),

$$\sigma_2'(0^+) = 0 \leq 0 \quad \text{and} \quad \sigma_2(T) > \sigma_2(\gamma) = u(\gamma) = C.$$

Therefore, σ_2 satisfies conditions (9.4.5)–(9.4.7) and so, σ_2 is an upper solution of (9.2.1), (9.4.1).

Step 3: Existence of a solution u_T :

We have found a pair of lower and upper solutions which clearly satisfy that

$$\sigma_1(t) \leq \sigma_2(t), \quad t \in [0, T] \text{ for each } T > \gamma.$$

As a consequence, Theorem 9.4.6 ensures the existence of a solution u_T of problem (9.2.1), (9.4.1) such that

$$L_0 \leq u_T(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Since $\sigma_2(0) = u_T(0) = L_0$, u satisfies (9.1.2) with $u_0 = L_0$.

Finally, since $f(\phi)$ is bounded on \mathbb{R} , u_T can be extended to the interval $[0, \infty)$ as a solution of equation (9.2.1).

Step 4: $u_T \geq L_0$ on $[0, \infty)$:

Analogously to the proof of Lemma 9.3.3, if we define

$$a := \inf \{ \tau > 0 : L_0 < u_T(\tau) < 0 \}$$

and

$$\theta := \sup \{ \tau > a : L_0 < u_T(\tau) < 0 \},$$

it occurs that $u_T'(a) = 0$ and $u_T'(t) > 0$ for $t \in (a, \theta)$.

In particular, since $u_T(T) = C < 0$, this implies that $a \in [0, T)$ and $\theta > T$.

Now, we have two possibilities:

- (i) First, if $\theta = \infty$, then $u'_T(t) > 0$ for $t \in (a, \infty)$ and this implies that $u_T > L_0$ on (a, ∞) . Thus, $u_T \geq L_0$ on $[0, \infty)$.
- (ii) On the other hand, if $\theta < \infty$, it occurs that θ is the first zero of u_T on (a, ∞) and $u'_T(\theta) > 0$.

Now, if $u'_T > 0$ on (θ, ∞) , then $u_T > L_0$ on (a, ∞) and the result holds.

On the contrary, there exists $\theta_1 > \theta$ such that $u'_T(\theta_1) = 0$ and $u'_T > 0$ on (θ, θ_1) and, from Lemma 9.3.3, $u_T(\theta_1) \in (0, L]$. Again, we may consider two possibilities: either $u_T > 0$ on (θ_1, ∞) and the proof is finished, or there exists $\theta_2 > \theta_1$ such that $u_T(\theta_2) = 0$ and $u_T > 0$ on $[\theta_1, \theta_2)$. In the second case, applying recursively Lemmas 9.2.3 and 9.2.4, we would conclude that $\bar{B} < u_T(t) < L$ for $t \in (\theta_2, \infty)$ and thus the result holds.

Therefore, we conclude that u_T is a solution of problem (9.2.1), (9.1.2) with $u_0 = L_0$ and satisfies (9.4.11). \square

Theorem 9.4.8 (Existence of escape solutions of problem (9.2.1), (9.1.2) II). *Let conditions (B_1) – (B_4) , (9.2.3) and (9.2.4) hold. Then there exist infinitely many escape solutions of problem (9.2.1), (9.1.2) with not necessary different starting values in $[L_0, \bar{B})$.*

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 9.2.10, there exists a solution u_n of problem (9.2.1), (9.1.2) with $u_0 = B_n$.

By Lemma 9.2.1, there exists a maximal $a_n > 0$ such that $u'_n > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then

$$u_n(t) < L \quad \text{and} \quad u'_n(t) > 0, \quad t \in (0, b_n).$$

Due to Lemmas 9.2.1 and 9.2.5, there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$.

From the sequence $\{B_n\}_{n=1}^\infty \subset (L_0, C)$, we get a sequence $\{u_n\}_{n=1}^\infty$ of solutions of problem (9.2.1), (9.1.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^\infty$. Assume that $\lim_{n \rightarrow \infty} B_n = L_0$.

Now, integrating equation (9.2.1) we get the equivalent form of problem (9.2.1), (9.1.2) for u_n

$$u_n(t) = B_n + \int_0^t \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}(\phi(u_n(\tau))) d\tau \right) ds, \quad t \in [0, \infty). \quad (9.4.13)$$

We will prove that the sequence $\{u_n\}_{n=1}^\infty$ is uniformly bounded on $[0, \beta]$ for all $\beta > 0$. Indeed, for $t \in [0, \beta]$,

$$\begin{aligned} |u_n(t)| &\leq |L_0| + \int_0^t \left| \phi^{-1}(\widetilde{M} \varphi(s)) \right| ds \leq |L_0| + \int_0^t \phi^{-1}(\widetilde{M} \beta) ds \\ &\leq |L_0| + \beta \phi^{-1}(\widetilde{M} \beta) \equiv K_\beta, \end{aligned}$$

where φ is defined in (9.2.7) and \widetilde{M} is from (9.2.9).

Moreover, as a consequence of Lemma 9.2.9, we know that the sequence of derivatives $\{u'_n\}_{n=1}^\infty$ is uniformly bounded. Therefore, the sequence $\{u_n\}_{n=1}^\infty$ is equicontinuous.

Therefore, by Ascoli-Arzelà's Theorem (Theorem 1.2.2), there exists a subsequence of $\{u_n\}_{n=1}^\infty$ which converges locally uniformly on $[0, \infty)$ to a continuous function u . For the sake of simplicity, we denote this subsequence also as $\{u_n\}_{n=1}^\infty$.

In particular, if we take the limit when t goes to infinity on equation (9.4.13), since the convergence is locally uniform, we obtain that u satisfies the following

$$u(t) = L_0 + \int_0^t \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) \widetilde{f}(\phi(u(\tau))) d\tau \right) ds, \quad t \in [0, \infty),$$

and therefore, u is a solution of problem (9.2.1), (9.1.2) for $u_0 = L_0$.

Now, we distinguish three different cases:

(i) $u \equiv L_0$:

In this case, $\lim_{n \rightarrow \infty} \gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded.

By Lemma 9.3.2 there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (9.2.1), (9.1.2). We have $u_{n_0}(0) = B_{n_0} > L_0$.

Now consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^\infty$. By Lemma 9.3.2 there exists $n_1 \in \mathbb{N}$ such that u_{n_1} is an escape solution of problem (9.2.1), (9.1.2) such that $u_{n_1}(0) = B_{n_1} > L_0$.

We repeat this procedure and we obtain the sequence $\{u_{n_k}\}_{k=0}^\infty$ of escape solutions of problem (9.2.1), (9.1.2) with starting values in (L_0, \bar{B}) .

(ii) $u \not\equiv L_0$ is not an escape solution:

In this case, we define $\widetilde{B}_n = L_0$ for all $n \in \mathbb{N}$ and consider γ defined in (9.4.12). Now, we can take an unbounded sequence $\{\widetilde{\gamma}_n\}_{n=1}^\infty$ such that $\widetilde{\gamma}_n > \gamma$ for all $n \in \mathbb{N}$.

By Lemma 9.4.7, for all $n \in \mathbb{N}$ there exists a solution \widetilde{u}_n of problem (9.2.1), (9.1.2) with $u_0 = \widetilde{B}_n$ such that

$$\widetilde{u}_n(\widetilde{\gamma}_n) = C, \quad \widetilde{u}_n(t) \geq L_0, \quad t \in [0, \infty).$$

Therefore, we have a sequence of solutions $\{\tilde{u}_n\}_{n=1}^{\infty}$ satisfying the conditions of Lemma 9.3.5 and so, this sequence contains an escape solution \tilde{u}_{n_0} of (9.2.1), (9.1.2) with $u_0 = L_0$.

As in the previous case, we could consider now the unbounded subsequence $\{\tilde{\gamma}_n\}_{n=n_0+1}^{\infty}$ and repeat the procedure from (i). This way we obtain a sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (9.2.1), (9.1.2) with $u_0 = L_0$.

(iii) $u \not\equiv L_0$ is an escape solution:

In this case, we can argue as in (ii) and we also obtain a sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (9.2.1), (9.1.2) with $u_0 = L_0$.

Moreover, in this case, since the sequence $\{u_n\}_{n=0}^{\infty}$ converges locally uniformly to an escape solution of (9.2.1), (9.1.2), there must exist some n_0 such that u_n is also an escape solution for all $n \geq n_0$. As a consequence, we also obtain a sequence $\{u_n\}_{n=n_0}^{\infty}$ of escape solutions of problem (9.2.1), (9.1.2) with starting values in (L_0, \bar{B}) . \square

9.5. Unbounded Solutions

In this section, we discuss the existence of escape solutions of the original problem (9.1.1)–(9.1.2) and provide conditions which guarantee that an escape solution of such problem is unbounded.

Note that, when (B_1) – (B_4) , (9.2.3) and (9.2.4) are assumed, solutions of the original problem (9.1.1)–(9.1.2) and solutions of the auxiliary problem (9.2.1), (9.1.2) are related in the following way:

- Each solution of (9.2.1), (9.1.2) which is not an escape solution, is a bounded solution of the original problem (9.1.1)–(9.1.2) in $[0, \infty)$. This results from Lemma 9.2.8 and Lemma 9.3.1, where such solutions of (9.2.1), (9.1.2) satisfy

$$L_0 \leq u(t) \leq L, \quad t \in [0, \infty)$$

and, due to (9.2.2),

$$\tilde{f}(\phi(u(t))) = f(\phi(u(t))), \quad t \in [0, \infty).$$

- If u is an escape solution of the auxiliary problem (9.2.1), (9.1.2), i.e.

$$\exists c \in (0, \infty): u(t) \in [L_0, L), \quad t \in [0, c), \quad u(c) = L, \quad u'(c) > 0, \quad (9.5.1)$$

then u fulfils at once the auxiliary equation (9.2.1) and the original equation (9.1.1) on $[0, c]$. The restriction of u on $[0, c]$ can be extended as an escape

solution of problem (9.1.1)–(9.1.2) on some maximal interval $[0, b)$. This situation is represented in Figure 9.5.1.

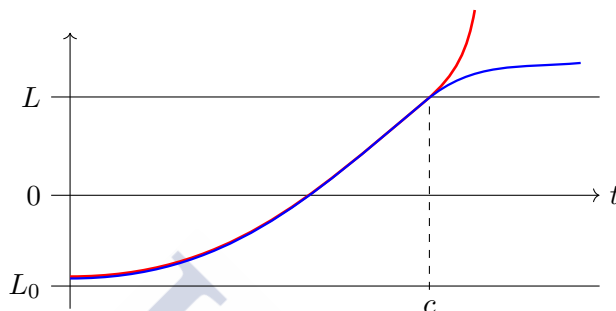


Figure 9.5.1: Relation between an escape solution of the auxiliary problem (9.2.1), (9.1.2) (function in red) and the original one (9.1.1)–(9.1.2) (in blue).

Therefore, we will search unbounded solutions of (9.1.1)–(9.1.2) in the set of escape solutions of (9.1.1)–(9.1.2) on $[0, b)$.

Since in general, an escape solution u of (9.1.1)–(9.1.2) on $[0, b)$ does not need to be unbounded (see Figure 9.5.2), we will derive some criteria for u to tend to infinity.

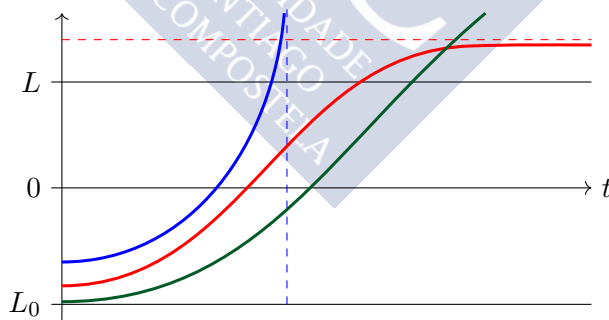


Figure 9.5.2: Various types of escape solutions of problem (9.1.1)–(9.1.2). In this case, $b < \infty$ for the solution in blue (which is unbounded) and $b = \infty$ for the solutions in green (which is also unbounded) and red (which is bounded).

Lemma 9.5.1. Assume that (B_1) – (B_4) hold. Let u be an escape solution of problem (9.1.1)–(9.1.2) on $[0, b)$. Then

$$u(t) > L, \quad u'(t) > 0, \quad t \in (c, b),$$

where c is from (9.5.1). Moreover, if $b < \infty$, then

$$\lim_{t \rightarrow b^-} u(t) = \infty.$$

Proof. Let u be an escape solution of problem (9.1.1)–(9.1.2) on $[0, b)$. Then

$$\exists c \in (0, \infty): u(t) \in [L_0, L], \quad t \in [0, c), \quad u(c) = L, \quad u'(c) > 0.$$

Assume that there exists $c_1 > c$ such that

$$u'(c_1) = 0, \quad u(t) > L, \quad u'(t) > 0 \quad \text{for } t \in (c, c_1).$$

Integrating equation (9.1.1) over $[c, c_1]$, dividing by $p(t)$ and using (B_1) , (B_3) and (B_4) , we get

$$\phi(u'(t)) = \frac{p(c) \phi(u'(c))}{p(t)} - \frac{1}{p(t)} \int_c^t p(s) f(\phi(u(s))) \, ds > 0, \quad t \in [c, c_1],$$

which contradicts that $u'(c_1) = 0$. Hence, $u(t) > L$ and $u'(t) > 0$ for $t \in (c, b)$.

Let $b < \infty$. Since $[0, b)$ is the maximal interval where the solution u is defined, u cannot be extended behind b . Therefore, since $u'(t) > 0$ for $t \in (c, b)$, it holds that $\lim_{t \rightarrow b^-} u(t) = \infty$ and thus, the solution u is unbounded. \square

Since all escape solutions of (9.2.1), (9.1.2) on $[0, b)$ which cannot be extended to the half-line $[0, \infty)$ are naturally unbounded, we continue our investigation about unboundedness of escape solutions defined on $[0, \infty)$. That is, we will assume from now on that $[0, b) = [0, \infty)$.

Theorem 9.5.2. Assume (B_1) – (B_4) hold and let

$$\lim_{t \rightarrow \infty} p(t) < \infty. \tag{9.5.2}$$

Let u be an escape solution of problem (9.1.1)–(9.1.2). Then

$$\lim_{t \rightarrow \infty} u(t) = \infty.$$

Proof. Let u be an escape solution of problem (9.1.1)–(9.1.2). Lemma 9.5.1 ensures that

$$u'(t) > 0, \quad t \in (c, \infty),$$

with c from (9.5.1), and so, there exists $\lim_{t \rightarrow \infty} u(t) \in (L, \infty]$. Due to (B_1) , (B_4) and (9.5.1),

$$p(c) \phi(u'(c)) =: c_0 \in (0, \infty).$$

Integrating now equation (9.1.1) from c to $t > c$, we get, by (B_3) and (B_4) , that

$$u(t) = L + \int_c^t \phi^{-1} \left(\frac{c_0}{p(s)} - \frac{1}{p(s)} \int_c^s p(\tau) f(\phi(u(\tau))) d\tau \right) ds > \int_c^t \phi^{-1} \left(\frac{c_0}{p(s)} \right) ds,$$

for $t \in (c, \infty)$.

Conditions (B_4) and (9.5.2) warrant that

$$\lim_{s \rightarrow \infty} \frac{c_0}{p(s)} \in (0, \infty)$$

and, by (B_1) ,

$$\int_1^\infty \phi^{-1} \left(\frac{c_0}{p(s)} \right) ds = \infty.$$

Therefore,

$$\lim_{t \rightarrow \infty} u(t) \geq \int_c^\infty \phi^{-1} \left(\frac{c_0}{p(s)} \right) ds = \infty,$$

which implies that the solution is unbounded. \square

Theorem 9.5.3. Assume (B_1) – (B_4) , (9.2.3) and

$$f(x) < 0 \quad \text{for } x > \phi(L). \quad (9.5.3)$$

Let u be an escape solution of problem (9.1.1)–(9.1.2). Then u is unbounded.

Proof. Let u be an escape solution of problem (9.1.1)–(9.1.2). Lemma 9.5.1 implies that $u' > 0$ on (c, ∞) and hence, there exists $\lim_{t \rightarrow \infty} u(t) \in (L, \infty]$. Assume on the contrary that

$$\lim_{t \rightarrow \infty} u(t) =: A \in (L, \infty). \quad (9.5.4)$$

Step 1: We prove that u' is bounded.

Assume that u' is unbounded. Then there exists a sequence $\{t_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and $\lim_{n \rightarrow \infty} u'(t_n) = \infty$.

Equation (9.1.1) has an equivalent form

$$\phi'(u'(t)) u''(t) + \frac{p'(t)}{p(t)} \phi(u'(t)) + f(\phi(u(t))) = 0, \quad t \in (0, \infty). \quad (9.5.5)$$

Choose $n \in \mathbb{N}$. Multiplying this equation by u' and integrating it from c to $t > c$, we obtain for $t = t_n$ that

$$\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n) = 0, \quad t_n \in [c, \infty), \quad (9.5.6)$$

where

$$\begin{aligned}\psi_1(t_n) &= \int_{u'(c)}^{u'(t_n)} x \phi'(x) \, dx, \\ \psi_2(t_n) &= \int_c^{t_n} \frac{p'(s)}{p(s)} \phi(u'(s)) u'(s) \, ds, \\ \psi_3(t_n) &= \int_L^{u(t_n)} f(\phi(x)) \, dx.\end{aligned}$$

Then $\psi_3(t_n) = F(u(t_n)) - F(L)$, where

$$F(x) := \int_0^x f(\phi(s)) \, ds, \quad x \in \mathbb{R}.$$

Due to (B_1) and (9.5.3), $F(x)$ is decreasing for $x > \phi(L)$. Since u is increasing on (c, ∞) , $F(u(t_n))$ is decreasing for $t_n \in (c, \infty)$ and $\lim_{n \rightarrow \infty} F(u(t_n)) = F(A)$. According to (9.5.4),

$$\lim_{n \rightarrow \infty} \psi_3(t_n) \in (-\infty, 0),$$

and, by (B_1) and (B_4) ,

$$\lim_{n \rightarrow \infty} \psi_1(t_n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \psi_2(t_n) > 0.$$

Hence, letting $n \rightarrow \infty$ in (9.5.6), we obtain

$$0 = \lim_{n \rightarrow \infty} (\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n)) = \infty,$$

which is a contradiction. So, u' is bounded.

Step 2: We will prove that $\lim_{t \rightarrow \infty} u(t) = \infty$.

Since u' is bounded, letting $t \rightarrow \infty$ in (9.5.5) and using (9.2.3), (9.5.3) and (9.5.4), we get

$$\lim_{t \rightarrow \infty} \phi'(u'(t)) u''(t) = -f(\phi(A)) > 0.$$

Since $\phi'(u'(t)) > 0$ for $t > c$, there exists $\tau > c$ such that $u''(t) > 0$ for $t \geq \tau$. Therefore, u' is increasing on $[\tau, \infty)$ and there exists $\lim_{t \rightarrow \infty} u'(t) > 0$, which contradicts $\lim_{t \rightarrow \infty} u(t) = A < \infty$. Thus, the solution is unbounded. \square

The following corollary can be deduced from the proof of Theorem 9.5.3.

Corollary 9.5.4. Assume conditions (B_1) – (B_4) and (9.2.3) and let u be a solution of problem (9.1.1)–(9.1.2). If u satisfies that

$$\lim_{t \rightarrow \infty} u(t) =: A \in (L, \infty),$$

then $f(\phi(A)) = 0$.

Remark 9.5.5. Note that, in previous corollary, $f(\phi(A)) = 0$ is equivalent with the fact that $u(t) \equiv A$ is a solution of equation (9.1.1).

For $f \equiv 0$ on $(\phi(L), \infty)$, we are able to find necessary and sufficient condition for the unboundedness of escape solutions of problem (9.1.1)–(9.1.2).

Theorem 9.5.6. Assume (B_1) – (B_4) ,

$$f(x) \equiv 0 \quad \text{for } x > \phi(L) \quad (9.5.7)$$

and

$$\phi(ab) = \phi(a)\phi(b), \quad a, b \in (0, \infty). \quad (9.5.8)$$

Let u be an escape solution of problem (9.1.1)–(9.1.2). Then u is unbounded if and only if

$$\int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds = \infty. \quad (9.5.9)$$

If we replace condition (9.5.8) by

$$\phi(ab) \leq \phi(a)\phi(b), \quad a, b \in (0, \infty), \quad (9.5.10)$$

then (9.5.9) implies that u is unbounded.

Proof. Let u be an escape solution of problem (9.1.1)–(9.1.2). Then, according to Lemma 9.5.1, $u' > 0$ on (c, ∞) . Thus there exists $t_0 > c$ such that $u(t_0) > L$, $u'(t) > 0$ for $t \in [t_0, \infty)$. Therefore, there exists

$$\lim_{t \rightarrow \infty} u(t) \in (L, \infty].$$

Using (9.5.8), we obtain

$$\begin{aligned} \phi^{-1}(a)\phi^{-1}(b) &= \phi^{-1}(\phi(\phi^{-1}(a))\phi^{-1}(b)) = \phi^{-1}(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))) \\ &= \phi^{-1}(ab), \quad a, b \in (0, \infty). \end{aligned} \quad (9.5.11)$$

Due to (B_1) , (B_4) and (9.5.7),

$$p(t_0)\phi(u'(t_0)) =: c_0 \in (0, \infty) \quad \text{and} \quad f(\phi(u(t))) = 0 \quad \text{for } t \in [t_0, \infty).$$

Thus, integrating equation (9.1.1) from t_0 to $t > t_0$ and using (9.5.11), we get

$$\begin{aligned} u(t) &= u(t_0) + \int_{t_0}^t \phi^{-1} \left(\frac{c_0}{p(s)} \right) ds = u(t_0) \\ &\quad + \phi^{-1}(c_0) \left(\int_1^t \phi^{-1} \left(\frac{1}{p(s)} \right) ds - \int_1^{t_0} \phi^{-1} \left(\frac{1}{p(s)} \right) ds \right), \quad t \in (t_0, \infty). \end{aligned}$$

Letting $t \rightarrow \infty$ here, we get the equivalence.

Now, let us consider (9.5.10) instead of (9.5.8) and assume that (9.5.9). Then we continue analogously and obtain

$$\begin{aligned} \phi^{-1}(a) \phi^{-1}(b) &= \phi^{-1}(\phi(\phi^{-1}(a) \phi^{-1}(b))) \leq \phi^{-1}(\phi(\phi^{-1}(a)) \phi(\phi^{-1}(b))) \\ &= \phi^{-1}(ab), \end{aligned}$$

with $a, b \in (0, \infty)$, and

$$\begin{aligned} u(t) &= u(t_0) + \int_{t_0}^t \phi^{-1} \left(\frac{c_0}{p(s)} \right) ds \geq u(t_0) \\ &\quad + \phi^{-1}(c_0) \left(\int_1^t \phi^{-1} \left(\frac{1}{p(s)} \right) ds - \int_1^{t_0} \phi^{-1} \left(\frac{1}{p(s)} \right) ds \right), \quad t \in (t_0, \infty). \end{aligned}$$

We let $t \rightarrow \infty$ here and obtain that if (9.5.9), then the solution is unbounded. \square

9.6. Main Results and Examples

In this section, we first present the existence results about unbounded solutions of the original problem (9.1.1)–(9.1.2) in case that ϕ^{-1} and f are Lipschitz continuous (see Theorems 9.6.1, 9.6.3 and 9.6.5). Each of these theorems is afterwards illustrated by an example which is chosen in such a way that only this theorem is applicable, while none of the remaining two theorems can be used for this example.

Then, in Theorems 9.6.7, 9.6.9 and 9.6.11, we present the main existence results about unbounded solutions of the original problem (9.1.1)–(9.1.2) provided ϕ^{-1} and f do not need to be Lipschitz continuous. The illustration by examples is done as in the previous case and shows that none of these theorems is included in any of the two remaining ones.

In the whole section, we assume that (due to Definition 9.1.1) for each $n \in \mathbb{N}$, $[0, b_n) \subset [0, \infty)$ is a maximal interval such that a function u_n satisfies equation (9.1.1) for every $t \in (0, b_n)$.

Theorem 9.6.1. *Assume that conditions (B_1) – (B_4) , (9.2.3), (9.2.4), (9.2.10), (9.2.11) and (9.5.2) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with different starting values in (L_0, \bar{B}) , $n \in \mathbb{N}$.*

Proof. By Theorem 9.4.1, there exist infinitely many escape solutions u_n of problem (9.2.1), (9.1.2) with starting values in (L_0, \bar{B}) . Let us choose $n \in \mathbb{N}$. Then

$$\exists c_n \in (0, \infty): u_n(t) \in (L_0, L), \quad t \in [0, c_n), \quad u_n(c_n) = L, \quad u'_n(c_n) > 0.$$

Consider the restriction of u_n to $[0, c_n]$. Then there exists $b_n > c_n$ such that u_n can be extended as a solution of problem (9.1.1)–(9.1.2) on $[0, b_n]$. If $b_n < \infty$, then, due to Lemma 9.5.1,

$$\lim_{t \rightarrow b_n^-} u_n(t) = \infty,$$

so u_n is unbounded. If $b_n = \infty$, then Theorem 9.5.2 yields

$$\lim_{t \rightarrow \infty} u_n(t) = \infty,$$

that is u_n is unbounded, as well. □

Example 9.6.2. Consider problem (9.1.1)–(9.1.2) with

$$\begin{aligned} \phi(x) &= \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}, \\ f(x) &= \begin{cases} x(x + \sinh 4)(\sinh 1 - x), & x \in [-\sinh 4, \sinh 1], \\ \cos(x - \sinh 1) - 1, & x > \sinh 1, \end{cases} \\ p(t) &= \arctan t \quad \text{or} \quad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad t \in [0, \infty). \end{aligned}$$

Here $L_0 = -4$, $L = 1$, $\phi^{-1}(x) = \operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$. These functions p satisfy (B_4) , (9.5.2) and

$$\lim_{t \rightarrow \infty} \frac{(\arctan t)'}{\arctan t} = \lim_{t \rightarrow \infty} \frac{\frac{1}{t^2+1}}{\arctan t} = 0, \quad \lim_{t \rightarrow \infty} \frac{(\tanh t)'}{\tanh t} = \lim_{t \rightarrow \infty} \frac{\frac{1}{\cosh^2 t}}{\tanh t} = 0,$$

that is, (9.2.3) holds, as well. Functions ϕ and f fulfil (B_1) – (B_3) .

Moreover, $0 < L < -L_0$, ϕ is odd and

$$\begin{aligned} \tilde{F}(L_0) &= \int_0^{-4} \phi(s) (\phi(s) + \sinh 4) (\sinh 1 - \phi(s)) \, ds \\ &= \int_0^4 \phi(s) (\sinh 4 - \phi(s)) (\sinh 1 + \phi(s)) \, ds \\ &> \int_0^1 \phi(s) (\sinh 4 - \phi(s)) (\sinh 1 + \phi(s)) \, ds \\ &> \int_0^1 \phi(s) (\phi(s) + \sinh 4) (\sinh 1 - \phi(s)) \, ds = \tilde{F}(L), \end{aligned}$$

thus, (9.2.4) holds. Since f and ϕ^{-1} are Lipschitz continuous, conditions (9.2.10) and (9.2.11) are valid, too.

We have fulfilled all assumptions of Theorem 9.6.1. Since f has isolated zeros on $(\sinh 1, \infty)$, we cannot use neither Theorem 9.6.3 nor Theorem 9.6.5 here.

In the same way as in the proof of Theorem 9.6.1, we can prove the following Theorems 9.6.3 or 9.6.5, if we use in the proof Theorems 9.5.3 or 9.5.6, respectively, instead of Theorem 9.5.2.

Theorem 9.6.3. *Let (B_1) – (B_4) , (9.2.3), (9.2.4), (9.2.10), (9.2.11) and (9.5.3) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with different starting values in (L_0, \bar{B}) , $n \in \mathbb{N}$.*

Example 9.6.4. *Let us consider problem (9.1.1)–(9.1.2) with*

$$\begin{aligned}\phi(x) &= \ln(|x| + 1) \operatorname{sign} x, \quad x \in \mathbb{R}, \\ f(x) &= x(x + \ln 4)(\ln 2 - x), \quad x \in [-\ln 4, \infty), \\ p(t) &= t^\beta, \quad \beta > 0, \quad t \in [0, \infty).\end{aligned}$$

Here $L_0 = -3$, $L = 1$ and $\phi^{-1}(x) = (e^{|x|} - 1) \operatorname{sign} x$.

We can easily check that ϕ , f and p satisfy (B_1) – (B_4) , (9.2.3) and (9.5.3). In addition, $0 < L < -L_0$, ϕ is odd and we can show, similarly to Example 9.6.2, that (9.2.4) holds.

The Lipschitz continuity of f and ϕ^{-1} yields (9.2.10) and (9.2.11). Thus, we can apply Theorem 9.6.3 here.

Since $\lim_{t \rightarrow \infty} t^\beta = \infty$ and $f(x) < 0$ for $x > \ln 2$, we can not use neither Theorem 9.6.1 nor Theorem 9.6.5.

Theorem 9.6.5. *Assume that (B_1) – (B_4) , (9.2.3), (9.2.4), (9.2.10), (9.2.11), (9.5.7), (9.5.10) and (9.5.9) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with different starting values in (L_0, \bar{B}) , $n \in \mathbb{N}$.*

Example 9.6.6. *Consider problem (9.1.1)–(9.1.2) with*

$$\begin{aligned}\phi(x) &= x, \quad x \in \mathbb{R}, \\ p(t) &= \sqrt{t}, \quad t \in [0, \infty), \\ f(x) &= \begin{cases} x^3(x - \phi(L_0))(\phi(L) - x), & x \in [\phi(L_0), \phi(L)], \\ 0, & x > \phi(L), \end{cases} \quad 0 < L < -L_0.\end{aligned}$$

Functions ϕ , f , p and $\phi^{-1}(x) = x$ satisfy (B_1) – (B_4) , (9.2.3), (9.2.10), (9.2.11), (9.5.7), (9.5.8) and consequently, (9.5.10).

Since $f(\phi(x)) = f(x)$ and $L < -L_0$, we have $\tilde{F}(L) < \tilde{F}(L_0)$ and (9.2.4) holds. In addition,

$$\int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds = \int_1^\infty \frac{1}{\sqrt{s}} ds = \infty.$$

We have satisfied all assumptions of Theorem 9.6.5.

Since $\lim_{t \rightarrow \infty} \sqrt{t} = \infty$ and $f(x) < 0$ for $x > \ln 2$, we cannot use neither Theorem 9.6.1 nor Theorem 9.6.3.

Now, applying Theorem 9.4.8 instead of Theorem 9.4.1, we get as before the existence results about unbounded solutions in each case, where ϕ^{-1} and f do not have to be Lipschitz continuous.

Theorem 9.6.7. *Let (B_1) – (B_4) , (9.2.3), (9.2.4) and (9.5.2) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 9.6.8. *Let us consider problem (9.1.1)–(9.1.2) with $0 < L < -L_0$*

$$\phi(x) = |x|^\alpha \operatorname{sign} x, \quad \alpha > 1, \quad x \in \mathbb{R},$$

$$f(x) = \begin{cases} \sqrt{|x|} \operatorname{sign} x (x - \phi(L_0)) (\phi(L) - x), & x \in [\phi(L_0), \phi(L)], \\ (\phi(L) - x)(\phi(2L) - x), & x \in (\phi(L), \phi(2L)), \\ 0, & x \geq \phi(2L), \end{cases}$$

$$p(t) = \arctan t \quad \text{or} \quad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad t \in [0, \infty).$$

According to Example 9.6.2, functions p satisfy (B_4) , (9.2.3) and (9.5.2). Functions ϕ and f fulfil (B_1) – (B_3) . Since f is continuous, $0 < L < -L_0$ and ϕ is a continuous and odd function, (9.2.4) holds, too.

We have verified all assumptions of Theorem 9.6.7.

The form of f implies that neither Theorem 9.6.9 nor Theorem 9.6.11 can be applied.

Theorem 9.6.9. *Assume that (B_1) – (B_4) , (9.2.3), (9.2.4) and (9.5.3) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 9.6.10. Consider problem (9.1.1)–(9.1.2) with

$$\phi(x) = x^3, \quad x \in \mathbb{R},$$

$$f(x) = \sqrt[3]{x}(x+8)(1-x), \quad x \in [-8, \infty),$$

$$p(t) = t^\beta, \quad \beta > 0, \quad t \in [0, \infty).$$

Here $L_0 = -2$, $L = 1$, $\phi^{-1}(x) = \sqrt[3]{x}$. It is easy to see that ϕ , f and p fulfil (B_1) – (B_4) , (9.2.3) and (9.5.3). Further,

$$\tilde{F}(L_0) = \int_0^{-2} s(s^3+8)(1-s^3) \, ds = \frac{144}{5}$$

and

$$\tilde{F}(L) = \int_0^1 s(s^3+8)(1-s^3) \, ds = \frac{99}{40}.$$

So, $\tilde{F}(L_0) > \tilde{F}(L)$ which yields (9.2.4). Therefore, we can apply Theorem 9.6.9 here.

Since $\lim_{t \rightarrow \infty} t^\beta = \infty$ and $f(x) < 0$ for $x > 1$, we cannot use neither Theorem 9.6.7 nor Theorem 9.6.11.

Theorem 9.6.11. Let (B_1) – (B_4) , (9.2.3), (9.2.4), (9.5.7), (9.5.10) and (9.5.9) hold. Then there exist infinitely many unbounded solutions u_n of problem (9.1.1)–(9.1.2) on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.

Example 9.6.12. Let us consider problem (9.1.1)–(9.1.2) with

$$\phi(x) = |x|^\alpha \operatorname{sign} x, \quad \alpha > 1, \quad x \in \mathbb{R},$$

$$p(t) = t^\beta, \quad \beta \in (0, \alpha], \quad t \in [0, \infty),$$

$$f(x) = \begin{cases} \sqrt[3]{x}(x - \phi(L_0))(\phi(L) - x), & x \in [\phi(L_0), \phi(L)], \\ 0, & x > \phi(L), \end{cases} \quad 0 < L < -L_0.$$

Functions ϕ , f and p satisfy (B_1) – (B_4) , (9.2.3), (9.5.7), (9.5.8) and consequently, (9.5.10). Moreover, $0 < L < -L_0$ and ϕ is odd function which yields (9.2.4). Furthermore,

$$\phi^{-1}(x) = x^{\frac{1}{\alpha}} \quad \text{for } x > 0$$

and

$$\int_1^\infty \phi^{-1} \left(\frac{1}{p(s)} \right) \, ds = \int_1^\infty s^{-\frac{\beta}{\alpha}} \, ds = \infty,$$

that is, we have verified all assumptions of Theorem 9.6.11.

Since $\lim_{t \rightarrow \infty} t^\beta = \infty$ and $f(x) = 0$ for $x > \phi(L)$, neither Theorem 9.6.7 nor Theorem 9.6.9 are applicable.

It is clear that every unbounded solution of problem (9.1.1)–(9.1.2) is an escape solution. According to the proofs of above theorems, we can formulate also the reverse assertion.

Corollary 9.6.13. *Assume all assumptions of Theorem 9.6.1 or 9.6.3 or 9.6.5 or 9.6.7 or 9.6.9 or 9.6.11. Then each escape solution of problem (9.1.1)–(9.1.2) is unbounded.*





Resumen

La presente Tesis, englobada bajo el título “*Ecuaciones diferenciales no lineales en dominios acotados y no acotados*”, contiene la práctica totalidad del trabajo desarrollado por la autora en los últimos años.

Se encuentra dividida en dos partes diferenciadas: la primera de ellas, que consta de seis capítulos, comprende el estudio de problemas de frontera definidos en intervalos acotados, así como el caso más general de considerar ecuaciones integrales de Hammerstein. La segunda parte, compuesta por tres capítulos, se centra en el estudio de problemas diferenciales e integrales definidos sobre dominios no acotados.

Cabe indicar además que, pese a que en el título se mencionan únicamente las ecuaciones diferenciales no lineales, los primeros capítulos de la Tesis se dedicarán al estudio de problemas de frontera lineales. Esto es así puesto que las propiedades de tales problemas, y particularmente las de la función de Green asociada, determinarán la mejor forma de abordar la búsqueda de soluciones de problemas no lineales.

Se incluye a continuación un breve resumen de los resultados principales tratados en cada capítulo.

Capítulo 1: Resultados Preliminares

Con el objetivo de escribir un trabajo autocontenido, este capítulo está dedicado a recopilar los resultados previos que se usarán a lo largo de la presente Tesis.

En primer lugar, en la Sección 1.1 se introducen la definición y las propiedades de la función de Green. Como veremos, esta función resulta una herramienta muy útil para estudiar problemas diferenciales tanto lineales como no lineales. Esto se debe al hecho de que todo problema diferencial se puede transformar en otro integral equivalente, cuyo núcleo es precisamente la función de Green.

De este modo, el problema de encontrar soluciones de problemas diferenciales llevará de forma natural al marco más general de encontrar puntos fijos de operadores integrales. Es en este contexto en el cual los resultados que aseguran la existencia de puntos fijos de operadores compactos arbitrarios definidos en espacios de Banach adquieren una gran importancia. Algunos de estos resultados se recogen en la Sec-

ción 1.2, entre ellos el bien conocido Teorema de punto fijo de Schauder y la teoría clásica del índice de punto fijo (la cual, siguiendo la línea de [64], se introducirá para conjuntos abiertos arbitrarios, los cuales podrían ser no acotados).

Finalmente, otra herramienta importante que usaremos en esta Tesis para estudiar propiedades de operadores lineales es la teoría espectral. En particular, la combinación de esta teoría con los resultados de índice de punto fijo, permitirá probar la existencia de soluciones de ciertos problemas integrales. Algunos resultados básicos de teoría espectral se recopilan en la Sección 1.3.

Capítulo 2: Funciones de Green y Teoría Espectral para Problemas de Frontera de Orden Par

Este capítulo contiene un estudio detallado de los problemas de frontera lineales de orden par. En particular, estudiaremos problemas asociados al siguiente operador bajo diversas condiciones de frontera:

$$L u(t) \equiv u^{(2n)}(t) + a_{2n-1}(t) u^{(2n-1)}(t) + \cdots + a_1(t) u'(t) + a_0(t) u(t), \quad t \in I \equiv [0, T],$$

donde $a_k : I \rightarrow \mathbb{R}$, $a_k \in L^\alpha(I)$, $\alpha \geq 1$, $k = 0, \dots, 2n-1$.

A partir de este operador definiremos otros dos, concretamente

$$\tilde{L} u(t) \equiv u^{(2n)}(t) + \hat{a}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{a}_{2n-2}(t) u^{(2n-2)}(t) + \cdots + \hat{a}_1(t) u'(t) + \tilde{a}_0(t) u(t), \quad t \in J \equiv [0, 2T],$$

donde \tilde{a}_{2k} , $k = 0, \dots, n-1$, es la extensión par de a_{2k} a J y \hat{a}_{2k+1} , $k = 0, \dots, n-1$ es la extensión impar de a_{2k+1} a J , y

$$\tilde{\tilde{L}} u(t) \equiv u^{(2n)}(t) + \hat{\hat{a}}_{2n-1}(t) u^{(2n-1)}(t) + \tilde{\tilde{a}}_{2n-2}(t) u^{(2n-2)}(t) + \cdots + \hat{\hat{a}}_1(t) u'(t) + \tilde{\tilde{a}}_0(t) u(t), \quad t \in [0, 4T],$$

donde $\tilde{\tilde{a}}_{2k}$ y $\hat{\hat{a}}_{2k+1}$, $k = 0, \dots, n-1$, son las extensiones par e impar al intervalo $[0, 4T]$ de \tilde{a}_{2k} y \hat{a}_{2k+1} , respectivamente.

La idea principal de este capítulo consiste en expresar la función de Green de problemas de Neumann, Dirichlet y mixtos asociados al operador L como suma de funciones de Green de problemas periódicos y antiperiódicos relativos a \tilde{L} . De esta

forma se prueban las siguientes igualdades

$$\begin{aligned} G_N[T](t, s) &= G_P[2T](t, s) + G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_D[T](t, s) &= G_P[2T](t, s) - G_P[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_{M_1}[T](t, s) &= G_A[2T](t, s) - G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \\ G_{M_2}[T](t, s) &= G_A[2T](t, s) + G_A[2T](2T - t, s), \quad \forall (t, s) \in I \times I, \end{aligned}$$

donde $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ y $G_{M_2}[T]$ denotan, respectivamente, las funciones de Green de los problemas de Neumann, Dirichlet y mixtos asociados al operador L . Análogamente, $G_P[2T]$ y $G_A[2T]$ denotan las funciones de Green de los problemas periódico y antiperiódico asociados al operador \tilde{L} .

Del mismo modo, se puede ver que todas las funciones de Green anteriores se expresan como combinación lineal de la relativa al problema periódico asociado a \tilde{L} evaluada en diferentes puntos.

Puesto que la función de Green es una herramienta fundamental para el estudio de problemas lineales y no lineales, poder relacionar de esta forma distintas funciones de Green permite establecer también una relación entre los distintos problemas, sus espectros y sus soluciones.

En primer lugar, las expresiones anteriores dan una conexión directa entre los espectros de los distintos problemas. En particular, deducimos varias descomposiciones de algunos espectros como unión de otros. Además, se obtiene una cierta relación de orden entre los primeros autovalores de cada problema.

Por otra parte, también deducimos que el signo constante de una función de Green implica el signo constante de otra.

Corolario 1 (Corollary 2.4.1). *Para cualesquiera coeficientes $a_0, \dots, a_{2n-1} \in L^1(I)$, se tienen las siguiente implicaciones:*

1. Si $G_P[2T] \leq 0$ en $J \times J$, entonces $G_N[T] \leq 0$ en $I \times I$.
2. Si $G_P[2T] \geq 0$ en $J \times J$, entonces $G_N[T] \geq 0$ en $I \times I$.
3. Si $G_N[2T] \leq 0$ en $J \times J$, entonces $G_N[T] \leq 0$ en $I \times I$.
4. Si $G_N[2T] \geq 0$ en $J \times J$, entonces $G_N[T] \geq 0$ en $I \times I$.
5. Si $G_D[2T] \leq 0$ en $J \times J$, entonces $G_{M_2}[T] \leq 0$ en $I \times I$.
6. Si $G_D[2T] \geq 0$ en $J \times J$, entonces $G_{M_2}[T] \geq 0$ en $I \times I$.

Con respecto al corolario anterior, cabe indicar que se puede mejorar para orden $n = 1$, lo cual se hace en el Capítulo 3. Por otra parte, se prueba en este capítulo que el recíproco de las Afirmaciones 1 y 2 del Corolario 1 se cumple si los coeficientes a_0, \dots, a_{2n-1} son contantes, mientras que el recíproco de las demás afirmaciones no es cierto ni siquiera en este caso para $n > 1$. Por otra parte, se da un contraejemplo para ver que el recíproco de 2 no es cierto en general para $n > 1$. Queda abierto el problema de ver si la Afirmación 1 es o no una equivalencia cuando $n > 1$.

Finalmente, en la Sección 2.5, asumiendo que una función de Green tiene signo constante, se obtienen desigualdades punto a punto entre otras dos funciones de Green distintas. Esto permite deducir que la solución del problema bajo ciertas condiciones de frontera es menor o igual en todo punto que la solución de otro problema en el que se considere el mismo operador pero condiciones de frontera distintas.

Los resultados de este capítulo se pueden encontrar en [31].

Capítulo 3: Ecuación de Orden Dos

En este capítulo se considera el problema estudiado en el Capítulo 2 en el caso particular de la ecuación de orden dos (es decir, se consideraría $n = 1$).

El motivo por el cual se estudia este caso de forma independiente al general es que al trabajar con ecuaciones diferenciales de orden dos es posible utilizar la teoría de Sturm-Liouville. Esta teoría, que no es válida para ecuaciones diferenciales de orden superior, proporciona propiedades de oscilación de las soluciones de las ecuaciones. Tal hecho permitirá obtener resultados más fuertes que los del capítulo anterior.

En este capítulo se estudian dos problemas distintos. En primer lugar, en la Sección 3.2, se estudia el problema asociado al operador de Hill

$$L u(t) \equiv u''(t) + a(t) u(t), \quad t \in I,$$

el cual es un caso particular del operador L considerado en el Capítulo 3 para $n = 1$ y $a_1 \equiv 0$. Cabe observar que el hecho de considerar $a_1 \equiv 0$ no supone una gran pérdida de generalidad de los resultados obtenidos puesto que cualquier ecuación diferencial de orden 2 de la forma

$$u''(t) + a_1(t) u'(t) + a_0(t) u(t) = 0,$$

puede transformarse en una ecuación de Hill mediante un cambio de variable adecuado, siempre y cuando los coeficientes a_0 y a_1 sean lo suficientemente regulares.

Los resultados obtenidos en esta sección son, pues, más potentes que los análogos obtenidos en el Capítulo 3. Un claro ejemplo de ello es el siguiente teorema en el cual se relaciona el signo constante de distintas funciones de Green.

Teorema 2 (Theorem 3.2.22). *Para todo $a \in L^1(I)$ se tienen las siguientes implicaciones:*

1. $G_P[2T] < 0$ en $J \times J$ si y solo si $G_N[T] < 0$ en $I \times I$. Esto es equivalente a $G_N[2T] < 0$ en $J \times J$.
2. $G_P[2T] > 0$ en $(0, 2T) \times (0, 2T)$ si y solo si $G_N[T] > 0$ en $(0, T) \times (0, T)$.
3. Si $G_N[2T] > 0$ en $(0, 2T) \times (0, 2T)$, entonces $G_N[T] > 0$ en $(0, T) \times (0, T)$.
4. Si $G_P[2T] < 0$ en $J \times J$, entonces $G_D[2T] < 0$ en $(0, 2T) \times (0, 2T)$.
5. Si $G_P[2T] > 0$ en $(0, 2T) \times (0, 2T)$, entonces $G_D[2T] < 0$ en $(0, 2T) \times (0, 2T)$.
6. Si $G_N[T]$ (o, equivalentemente, $G_P[2T]$) tiene signo constante en $I \times I$, entonces $G_D[T] < 0$ en $(0, T) \times (0, T)$, $G_{M_1}[T] < 0$ en $[0, T) \times [0, T)$ y $G_{M_2}[T] < 0$ en $(0, T) \times (0, T]$.
7. $G_D[2T] < 0$ en $(0, 2T) \times (0, 2T)$ si y solo si $G_{M_2}[T] < 0$ en $(0, T) \times (0, T]$.
8. Si $G_{M_2}[T] < 0$ en $(0, T) \times (0, T]$ o $G_{M_1}[T] < 0$ en $[0, T) \times [0, T)$, entonces $G_D[T] < 0$ en $(0, T) \times (0, T)$.

Del mismo modo, las desigualdades punto a punto entre distintas funciones de Green son también más precisas, lo cual supone una mayor precisión a la hora de comparar las soluciones de distintos problemas. Así, mientras en el capítulo anterior podíamos garantizar que la solución de un problema era menor en todo punto que la solución de otro, para este caso particular también podremos garantizar que las dos soluciones tienen signo constante.

Además, mientras que en el capítulo anterior solo podíamos establecer una relación de orden entre los primeros autovalores de cada problema, en este se establece una relación de alternancia entre todos los autovalores de todos los problemas.

Finalmente, para terminar esta sección, se consideran los criterios explícitos existentes en la literatura para garantizar el signo constante de la función de Green del problema periódico y, utilizando las relaciones entre las distintas funciones de Green, se adaptan a todos los demás problemas de frontera considerados.

Por otra parte, en la Sección 3.3 se considera una ecuación más general dada en forma autoadjunta, concretamente

$$(p u')'(t) + \bar{a}(t) u(t) = \bar{\sigma}(t), \quad \text{c. t. p. } t \in I,$$

con $p > 0$ c. t. p. $t \in I$, $\frac{1}{p} \in L^1(I)$ y \bar{a} y $\bar{\sigma}$ tales que $\bar{a} p^{\frac{\alpha-1}{\alpha}}$, $\bar{\sigma} p^{\frac{\alpha-1}{\alpha}} \in L^\alpha(I)$, para algún $\alpha \in [1, \infty]$.

Se demuestra en esta sección que la función de Green de cualquier problema de frontera asociado a la ecuación previa se puede expresar en términos de la función de Green asociada al operador de Hill con las mismas condiciones de frontera. Como consecuencia, todos los resultados obtenidos en la sección anterior se pueden adaptar en términos de este problema.

Este capítulo recoge resultados de [22] y [23].

Capítulo 4: Soluciones para Problemas de Frontera No Lineales de Orden Par con Funciones de Green de Signo Constante

En este capítulo se consideran por primera vez problemas de frontera no lineales. En particular, se considerarán problemas que sigan el siguiente esquema:

$$L u(t) = f(t, u(t)), \quad t \in I, \quad u \in X,$$

siendo L el operador general lineal de orden $2n$ definido en el Capítulo 3.

Por otra parte, consideraremos $X \subset W^{2n,1}(I)$ como un espacio de Banach que incluye las condiciones de frontera y en el cual L es no resonante.

En estas condiciones se tiene que las soluciones del problema de frontera anterior se corresponden con los puntos fijos en X del siguiente operador integral

$$L^{-1} u(t) = \int_0^T G[T](t, s) f(s, u(s)) \, ds,$$

siendo $G[T]$ la función de Green asociada.

El método utilizado para garantizar la existencia de puntos fijos de este operador integral es el de sub y sobresoluciones.

La novedad principal de nuestra aproximación frente a referencias previas presentes en la literatura es el hecho de que conseguimos garantizar la existencia de solución del problema mediante un par de sub y sobresoluciones de otro problema distinto (compuesto por el mismo operador sometido a condiciones de frontera diferentes). Esto será posible gracias a las relaciones punto a punto entre distintas funciones de Green probadas en los Capítulos 2 y 3.

Cabe comentar también que una de las hipótesis básicas de este capítulo es la del signo constante de las funciones de Green.

Los resultados de este capítulo se pueden ver en [31].

Capítulo 5: Soluciones Positivas para Problemas de Frontera No Lineales de Orden Dos con Funciones de Green de Signo No Constante

Este capítulo está dedicado a estudiar la existencia de soluciones de signo constante de un problema de frontera de orden dos asociado al operador de Hill en el caso en que, al contrario de lo que ocurría en el capítulo anterior, la función de Green cambie de signo.

La idea básica de este capítulo se fundamenta en el hecho de que, pese a que la función de Green cambie de signo, se puede asegurar que la integral de esta función multiplicada por la autofunción correspondiente al primer autovalor del problema es positiva.

Expondremos el siguiente razonamiento en términos del problema periódico, aunque resulta igualmente válido para cualquier otra condición de frontera.

Consideremos pues el siguiente problema periódico

$$\begin{cases} u''(t) + a(t)u(t) = f(t, u(t)), & t \in I, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

y sean G_P su función de Green asociada y v_P la autofunción correspondiente al primer autovalor. Entonces se tiene que

$$\int_0^T G_P(t, s) v_P(s) \, ds > 0, \quad \text{para todo } t \in I,$$

lo cual justifica que tiene sentido definir la siguiente constante:

$$\gamma = \inf_{t \in I} \frac{\int_0^T G_P^+(t, s) v_P(s) \, ds}{\int_0^T G_P^-(t, s) v_P(s) \, ds} (> 1).$$

Supongamos que se cumplen las siguientes hipótesis:

(H₁) $f: I \times [0, \infty) \rightarrow [0, \infty)$ satisface las condiciones de L¹-Carathéodory.

(H₂) Existen dos constantes positivas m y M tales que

$$m v_P(t) \leq f(t, x) \leq M v_P(t)$$

para todo $t \in I$ y $x \geq 0$. Además, estas constantes deben cumplir que $\frac{M}{m} \leq \gamma$.

(H₃) Existe un subintervalo $[c, d] \subset I$ tal que $\int_c^d G_P(t, s) \, dt \geq 0$, para todo $s \in I$ y $\int_c^d G_P(t, s) \, dt > 0$, para todo $s \in [c, d]$.

Entonces, si la función de Green cambia de signo, se demuestra que existe una solución del problema en el cono

$$K = \left\{ u \in \mathcal{C}(I, \mathbb{R}) : u \geq 0 \text{ en } I, \int_0^T u(s) \, ds \geq \sigma \|u\| \right\},$$

donde

$$\sigma = \frac{\eta}{\max_{t, s \in I} \{G_P(t, s)\}},$$

y

$$\eta = \min_{s \in [c, d]} \left\{ \int_c^d G_P(t, s) \, dt \right\} > 0.$$

Nótese que esta solución es no negativa.

Todos los resultados de este capítulo se recogen en [27].

Capítulo 6: Resultados de Existencia y Multiplicidad de Soluciones para Ecuaciones Generalizadas de Hammerstein con un Parámetro

En este capítulo estudiamos problemas integrales definidos en espacios de Banach que reciben el nombre de ecuaciones generalizadas de Hammerstein.

En particular, estudiamos la existencia y multiplicidad de puntos fijos del siguiente operador integral

$$\mathcal{T}u(t) = \lambda \int_0^T k(t, s) f(s, u(s), u'(s), \dots, u^{(m)}(s)) \, ds, \quad t \in I,$$

donde $\lambda > 0$ es un parámetro positivo, $k : I \times I \rightarrow \mathbb{R}$ es una función núcleo que verificará ciertas propiedades, m un entero positivo y $f : I \times \mathbb{R}^{m+1} \rightarrow [0, +\infty)$ es una función L^1 -Carathéodory.

Este capítulo generaliza varios resultados presentes en la literatura al pedir condiciones menos restrictivas de lo habitual sobre el núcleo.

En concreto, se pedirá que el núcleo y algunas de sus derivadas (no necesariamente todas) sean positivos únicamente en un subintervalo de I . Este subintervalo podría incluso llegar a ser degenerado, es decir, podría tratarse de un único punto.

Por otra parte, buscaremos núcleos para los cuales algunas de sus derivadas (de nuevo, no necesariamente todas) satisfagan las siguientes desigualdades:

$$\left| \frac{\partial^j k}{\partial t^j}(t, s) \right| \leq \phi_j(s) \quad \text{para todo } t \in [c_j, d_j] \text{ y c. t. p. } s \in I,$$

y

$$\frac{\partial^j k}{\partial t^j}(t, s) \geq \xi_j \phi_j(s) \text{ para todo } t \in [a_j, b_j] \text{ y c. t. p. } s \in I,$$

siendo ϕ_j funciones integrables y ξ_j constantes. Cabe comentar que los intervalos $[a_j, b_j]$ y $[c_j, d_j]$ deben tener intersección no vacía pero podrían ser distintos e, incluso, no comparables.

Bajo diversas hipótesis (véanse (H_1) – (H_7) en la Sección 6.2), podemos demostrar entonces la existencia de puntos fijos del operador integral considerado en el cono

$$K = \left\{ u \in C^m(I, \mathbb{R}) : \begin{array}{l} u^{(i)}(t) \geq 0, \quad t \in [m_i, n_i], \quad i \in J_0; \\ \min_{t \in [a_j, b_j]} u^{(j)}(t) \geq \xi_j \|u^{(j)}\|_{[c_j, d_j]}, \quad j \in J_1 \end{array} \right\},$$

donde

$$\|u^{(j)}\|_{[c_j, d_j]} := \max_{t \in [c_j, d_j]} |u^{(j)}(t)|,$$

$J \equiv \{0, 1, \dots, m\}$ y $J_1 \subset J_0 \subset J$, $J_1 \neq \emptyset$. Este tipo de conos, hasta donde la autora tiene conocimiento, es nuevo en la literatura.

En cuanto a las técnicas para demostrar la existencia de puntos fijos, se utilizan dos diferentes.

En primer lugar, en la Sección 6.3, se prueba la existencia de un punto fijo utilizando el índice de punto fijo para conjuntos abiertos arbitrarios (algunos de los cuales son no acotados).

Por otra parte, en la Sección 6.4 se dan resultados de existencia y multiplicidad de soluciones. Estos resultados se basan también en el índice de punto fijo, esta vez sobre conjuntos abiertos y acotados.

La diferencia principal entre ambas secciones es que las hipótesis que se le piden a la no linealidad f son diferentes y, de hecho, en la Sección 6.5 se muestran ejemplos en los que se ve que ambos métodos no son comparables.

A continuación, la Sección 6.6 presenta una aplicación de los resultados previos para garantizar la existencia de solución de problemas de Dirichlet de orden par arbitrario

$$\begin{cases} u^{(2n)}(t) = f(t, u(t), \dots, u^{(2n-1)}(t)), & t \in [0, 1], \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, \dots, n-1. \end{cases}$$

Este estudio generaliza los existentes en la literatura puesto que en este tipo de problemas se suele considerar que la función f depende únicamente de las derivadas de orden par, mientras que en este capítulo se admite la dependencia de cualquier derivada hasta orden $2n-1$.

Finalmente, la Sección 6.7 considera el caso particular del siguiente problema diferencial de orden tres

$$\begin{cases} -u^{(3)}(t) = \lambda f(t, u(t), u'(t), u''(t)), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta), \end{cases}$$

siendo $0 < \eta < 1$ y $1 < \alpha < \frac{1}{\eta}$ constantes dadas.

Los resultados de este capítulo se pueden encontrar en [32] y [102].

Capítulo 7: Problemas Multipunto Resonantes en la Semi-recta

En este capítulo consideraremos por primera vez un problema definido en un dominio no acotado.

En particular, probaremos la existencia de soluciones acotadas para el siguiente problema multipunto

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in [0, \infty), \\ u(0) = 0, & u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases}$$

siendo $\alpha_i > 0$ y $0 = \xi_1 < \dots < \xi_{m-1} < +\infty$. Asumiremos que los coeficientes α_i cumplen la siguiente condición

$$\sum_{i=1}^{m-1} \alpha_i = 1,$$

la cual implica que nos encontramos ante un problema resonante.

Para resolver este problema consideraremos otro modificado (el cual se construirá añadiendo nuevos términos a ambos lados de la ecuación) que será equivalente al primero y no resonante. Este problema modificado lo transformaremos en un problema integral cuyos puntos fijos se corresponderán con las soluciones del problema inicial. En concreto, el problema integral con el que trabajaremos será

$$Tu(t) = \int_0^\infty G(t, s) (f(s, u(s), u'(s)) + k u'(s) + M u(s)) \, ds,$$

donde G es la función de Green del problema

$$\begin{cases} u''(t) + k u'(t) + M u(t) = 0, & t \in [0, \infty), \\ u(0) = 0, & u'(+\infty) = \sum_{i=1}^{m-1} \alpha_i u'(\xi_i), \end{cases}$$

y k y M son dos constantes positivas que cumplen ciertas condiciones.

Además, el problema modificado satisfará otra propiedad importante: su función de Green estará en el espacio $L^1[0, \infty) \cap L^\infty[0, \infty)$. Esto permitirá que el operador integral sea compacto tanto si la no linealidad f satisface las condiciones L^1 o L^∞ -Carathéodory. Nótese que este hecho permite garantizar la existencia de solución para un mayor número de casos puesto que, al estar considerando en este capítulo un intervalo no acotado, los espacios $L^1[0, \infty)$ y $L^\infty[0, \infty)$ no son comparables.

Para probar la existencia de puntos fijos del operador integral utilizaremos el método de sub y sobresoluciones. En particular, para demostrar que el operador integral es compacto utilizaremos el criterio de compacidad dado en el Theorem 1, que involucra una cierta condición de equiconvergencia en infinito.

Los resultados de este capítulo se recogen en [103].

Capítulo 8: Existencia de Soluciones de Ecuaciones Integrales con Condiciones Asintóticas

En este capítulo estudiamos los puntos fijos de un operador integral definido sobre la recta real.

En general, la mayor dificultad cuando se intenta probar la existencia de puntos fijos de operadores integrales definidos en intervalos no acotados surge al demostrar que el operador considerado es compacto. Estos problemas se deben principalmente a la imposibilidad de utilizar el Teorema de Ascoli-Arzelà para probar la compacidad del operador.

La forma más habitual de resolver este problema consiste en utilizar un cierto criterio de compacidad (el cual hemos utilizado, precisamente, en el Capítulo 7), que se recoge en el Theorem 1, en la página 181.

En este capítulo presentamos un método alternativo que tendrá un doble beneficio: por una parte, nos permitirá utilizar el Teorema de Ascoli-Arzelà para probar la compacidad del operador. Por otra, nos garantizará que las soluciones encontradas tienen un cierto comportamiento asintótico.

Para ello, definimos un espacio de Banach que incluya esas propiedades asintóticas. En particular, para $n \in \mathbb{N}$, consideramos el espacio de las funciones reales de variable real que son de clase n y tienen límite en $\pm\infty$:

$$\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}) := \left\{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} : f|_{\mathbb{R}} \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}), \exists \lim_{t \rightarrow \pm\infty} f^{(j)}(t) \in \mathbb{R}, j = 0, \dots, n \right\},$$

siendo $\overline{\mathbb{R}} \equiv [-\infty, \infty]$. Se tiene que $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$, $n \in \mathbb{N}$ es un espacio de Banach con la norma

$$\|f\|_{(n)} := \sup \left\{ \|f^{(k)}\|_{\infty} : k = 0, \dots, n \right\}.$$

Entonces, dada una función $\varphi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^+)$, definimos el espacio de las φ -extensiones de clase n a infinito como sigue:

$$\tilde{\mathcal{C}}_\varphi^n \equiv \tilde{\mathcal{C}}_\varphi^n(\mathbb{R}, \mathbb{R}) = \left\{ f \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}) : \exists \tilde{f} \in \mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R}), f = \varphi \cdot \tilde{f}|_{\mathbb{R}} \right\}.$$

En particular, este espacio es de Banach con la norma inducida

$$\|f\|_\varphi := \|\tilde{f}\|_{(n)}, \quad f \in \tilde{\mathcal{C}}_\varphi^n,$$

de donde se deduce que los espacios $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ y $\tilde{\mathcal{C}}_\varphi^n$ son isomorfos.

De la existencia de dicho isomorfismo se deduce que, puesto que el Teorema de Ascoli-Arzelà se puede aplicar al espacio $\mathcal{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ (por ser $\overline{\mathbb{R}}$ compacto), entonces este teorema se puede aplicar también al espacio $\tilde{\mathcal{C}}_\varphi^n$.

Buscaremos pues puntos fijos de operadores integrales de la forma

$$Tu(t) := p(t) + \int_{-\infty}^{\infty} k(t, s) \eta(s) f(s, u(s)) \, ds$$

en el espacio de Banach $\tilde{\mathcal{C}}_\varphi^n$, para una cierta función φ que representará precisamente el comportamiento asintótico de las soluciones. Dicho de otro modo, que los puntos fijos del operador se encuentren en el espacio $\tilde{\mathcal{C}}_\varphi^n$ implicará que tales funciones se comporten asintóticamente de forma similar a φ .

En cuanto al método empleado para garantizar la existencia de puntos fijos, consideramos en este capítulo dos aproximaciones diferentes: la primera de ellas, desarrollada en la Sección 8.4 se basa en el índice de punto fijo en conos y presenta hipótesis bastante restrictivas sobre la función no lineal f .

Por otra parte, la segunda aproximación, analizada en la Sección 8.5, se basa en definir una serie de operadores lineales auxiliares y estudiar sus propiedades espectrales. En particular, si el radio espectral de estos operadores y ciertos límites obtenidos a partir de la función no lineal f satisfacen ciertas propiedades, será posible probar la existencia de puntos fijos. En este caso, las restricciones sobre la función f son mucho menos restrictivas que las impuestas por el método anterior, pero a expensas de pedir que el núcleo k satisfaga condiciones más fuertes.

Tal y como se muestra en el capítulo con ejemplos de los dos métodos, estos son no comparables.

Todos estos resultados se pueden ver en [33] y [34].

Capítulo 9: Soluciones no Acotadas de Problemas de Valores Iniciales Singulares con ϕ -Laplaciano

En este último capítulo se estudia un problema de valor inicial singular con ϕ -Laplaciano, prestando especial interés a la existencia de soluciones no acotadas del

mismo.

En este caso, al tratarse de un problema singular, no es posible construir un problema integral equivalente, tal y como se hace en los capítulos anteriores. Consecuentemente, las técnicas utilizadas en este capítulo difieren totalmente de las de consideradas hasta el momento.

En particular, consideraremos el siguiente problema no lineal:

$$\begin{cases} (p(t) \phi(u'(t)))' + p(t) f(\phi(u(t))) = 0, & t > 0, \\ u(0) = u_0, \quad u'(0) = 0, & u_0 \in [L_0, L]. \end{cases}$$

Comenzamos el capítulo definiendo tres tipos de soluciones posibles que podemos obtener. Así, si denotamos

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\},$$

diremos que

- Una solución u del problema es “oscilante” (damped) si $u_{\sup} < L$.
- Una solución u del problema es homoclínica si $u_{\sup} = L$.
- Una solución u del problema será “de escape” si $u_{\sup} > L$.

Puesto que tanto las soluciones oscilantes como las homoclínicas están acotadas, las soluciones no acotadas serán un subconjunto de las de escape. Esto motiva la división del capítulo en dos partes:

1. Búsqueda de condiciones para garantizar la existencia de soluciones de escape.
2. Búsqueda de condiciones necesarias o suficientes para garantizar que una solución de escape es no acotada.

Además, para la búsqueda de condiciones que aseguren la existencia de soluciones de escape tendremos que considerar dos casos diferenciados: el primero de ellos, en el que tanto f como ϕ^{-1} son funciones lipschitzianas, resulta bastante más sencillo puesto que en estas condiciones la unicidad de solución del problema está garantizada.

Por el contrario, el segundo caso (con f y ϕ^{-1} no lipschitzianas), presenta una serie de complicaciones derivadas de la no unicidad de solución. Para solventar estos problemas se considera el método de sub y sobresoluciones.

Estos dos casos presentan además otra diferencia importante en cuanto a los resultados obtenidos: mientras que en el primero se garantiza la existencia de una sucesión

de soluciones de escape que toman diferentes valores iniciales, en el segundo podría ocurrir que todas las soluciones tuvieran el mismo valor inicial L_0 .

Finalmente, en la última sección del capítulo se recopilan todos los resultados obtenidos y se enuncian explícitamente una serie de condiciones suficientes que aseguran la existencia de soluciones no acotadas del problema. Diversos ejemplos muestran que todos estos resultados son no comparables.

Todos los resultados de este capítulo se pueden ver en [131].



Bibliography

- [1] Abraham, F.F.: Homogeneous nucleation theory: the pretransition theory of vapor condensation. Academic Press, New York (1974)
- [2] Agarwal, R.P., O'Regan, D.: Infinite interval problems for differential, difference and integral equations. Springer Netherlands (2001)
- [3] Al Mosa, S., Elloe, P.: *Upper and lower solution method for boundary value problems at resonance*. Electronic Journal of Qualitative Theory of Differential Equations **40**, 1–13 (2016)
- [4] Amann, H.: *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*. SIAM rev. **18**(4), 620–709 (1976)
- [5] Bai, C., Fang, J.: *Existence of positive solutions for three-point boundary value problems at resonance*. Journal of Mathematical Analysis and Applications **291**(2), 538–549 (2004)
- [6] Bebernes, J., Gaines, R., Schmitt, K.: *Existence of periodic solutions for third and fourth order ordinary differential equations via coincidence degree*. Annales de la Société Scientifique de Bruxelles Série I **88**, 25–36 (1974)
- [7] Benmezai, A., Graef, J.R., Kong, L.: *Positive solutions for abstract Hammerstein equations and applications*. Communications in Mathematical Analysis **16**(1), 47–65 (2014)
- [8] Berestycki, H., Lions, P., Peletier, L.A.: *An ODE approach to the existence of positive solutions for semilinear problems in \mathbb{R}^N* . Indiana University Mathematics Journal **30**(1), 141–157 (1981)
- [9] Bernfeld, S.R., Lakshmikantham: An introduction to nonlinear boundary value problems, *Mathematics in Science and Engineering*, vol. 109. Academic Press, New York (1974)

- [10] Bonheure, D., Gomes, J.M., Sanchez, L.: *Positive solutions of a second-order singular ordinary differential equation*. Nonlinear Analysis: Theory, Methods & Applications **61**(8), 1383–1399 (2005)
- [11] Bugajewski, D., Zima, M.: *On the spectral radius of linearly bounded operators and existence results for functional-differential equations*. Discrete and Continuous Dynamical Systems **2003**, 147–155 (2003)
- [12] Burkotová, J., Hubner, M., Rachůnková, I., Weinmüller, E.B.: *Asymptotic properties of Kneser solutions to nonlinear second order ODEs with regularly varying coefficients*. Applied Mathematics and Computation **274**, 65–82 (2016)
- [13] Burkotová, J., Rachůnková, I., Rohleder, M., Stryja, J.: *Existence and uniqueness of damped solutions of singular IVPs with ϕ -Laplacian*. Electronic Journal of Qualitative Theory of Differential Equations **121**, 1–28 (2016)
- [14] Burkotová, J., Rohleder, M., Stryja, J.: *On the existence and properties of three types of solutions of singular IVPs*. Electronic Journal of Qualitative Theory of Differential Equations **29**, 1–25 (2015)
- [15] Cañizo, J.A.: *Derivación bajo la integral* (2004). URL <http://canizo.org/tex/dbi.pdf>
- [16] Cabada, A.: *The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems*. Journal of Mathematical Analysis and Applications **185**(2), 302–320 (1994)
- [17] Cabada, A.: *An overview of the lower and upper solutions method with nonlinear boundary value conditions*. Boundary Value Problems **2011** (2010)
- [18] Cabada, A.: *Green's functions in the theory of ordinary differential equations*. Springer Briefs in Mathematics (2014)
- [19] Cabada, A., Cid, J.Á.: *On the sign of the Green's function associated to Hill's equation with an indefinite potential*. Applied Mathematics and Computation **205**(1), 303–308 (2008)
- [20] Cabada, A., Cid, J.Á.: *Existence and multiplicity of solutions for a periodic Hill's equation with parametric dependence and singularities*. Abstract and Applied Analysis **2011**, 1–19 (2011)
- [21] Cabada, A., Cid, J.Á.: *On comparison principles for the periodic Hill's equation*. Journal of the London Mathematical Society **86**(1), 272–290 (2012)

- [22] Cabada, A., Cid, J.Á., López-Somoza, L.: *Green's functions and spectral theory for the Hill's equation*. Applied Mathematics and Computation **286**, 88–105 (2016)
- [23] Cabada, A., Cid, J.Á., López-Somoza, L.: *Maximum principles for the Hill's equation*. Academic Press, London (2018)
- [24] Cabada, A., Cid, J.Á., Máquez-Villamarín, B.: *Computation of Green's functions for boundary value problems with Mathematica*. Applied Mathematics and Computation **219**(4), 1919–1936 (2012)
- [25] Cabada, A., Cid, J.Á., Sanchez, L.: *Positivity and lower and upper solutions for fourth order boundary value problems*. Nonlinear Analysis: Theory, Methods & Applications **67**(5), 1599–1612 (2007)
- [26] Cabada, A., Cid, J.Á., Tvrdý, M.: *A generalized anti-maximum principle for the periodic one-dimensional p -Laplacian with sign-changing potential*. Nonlinear Analysis **72**(7), 3436–3446 (2010)
- [27] Cabada, A., Enguiça, R., López-Somoza, L.: *Positive solutions for second-order boundary-value problems with sign changing Green's functions*. Electronic Journal of Differential Equations **2017**(245), 1–17 (2017)
- [28] Cabada, A., Infante, G., Tojo, F.A.F.: *Nontrivial solutions of Hammerstein integral equations with reflections*. Boundary Value Problems **2013**(1), 1–22 (2013)
- [29] Cabada, A., Infante, G., Tojo, F.A.F.: *Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications*. Topological Methods in Nonlinear Analysis (2015)
- [30] Cabada, A., Infante, G., Tojo, F.A.F.: *Nonlinear perturbed integral equations related to nonlocal boundary value problems*. Fixed Point Theory **19**(1), 65–92 (2018)
- [31] Cabada, A., López-Somoza, L.: *Green's functions for even order boundary value problems*. arXiv preprint arXiv:1811.06116
- [32] Cabada, A., López-Somoza, L., Minhós, F.: *Existence, non-existence and multiplicity results for a third order eigenvalue three-point boundary value problem*. Journal of Nonlinear Sciences and Applications **10**, 5445–5463 (2017)

- [33] Cabada, A., López-Somoza, L., Tojo, F.A.F.: *Existence of solutions of integral equations defined in unbounded domains via spectral theory*. arXiv preprint arXiv:1811.06121
- [34] Cabada, A., López-Somoza, L., Tojo, F.A.F.: *Existence of solutions of integral equations with asymptotic conditions*. *Nonlinear Analysis: Real World Applications* **42**, 140–159 (2018)
- [35] Cabada, A., Minhós, F., Santos, A.I.: *Solvability for a third order discontinuous fully equation with nonlinear functional boundary conditions*. *Journal of Mathematical Analysis and Applications* **322**(2), 735–748 (2006)
- [36] Cabada, A., Nieto, J.J.: *Fixed points and approximate solutions for nonlinear operator equations*. *Journal of Computational and Applied Analysis* **113**(1–2), 17–25
- [37] Cabada, A., Pouso, R.L.: *On first order discontinuous scalar differential equations*. *Nonlinear Studies* **6**(2), 161–170 (1999)
- [38] Cabada, A., Saavedra, L.: *The eigenvalue characterization for the constant sign Green's functions of $(k, n - k)$ problems*. *Boundary Value Problems* **2016**(1), 1–44 (2016)
- [39] Cabada, A., Saavedra, L.: *Characterization of constant sign Green's function for a two-point boundary-value problem by means of spectral theory*. *Electronic Journal of Differential Equations* **2017**(146), 1–95 (2017)
- [40] Coppel, W.A.: *Disconjugacy*. *Lecture Notes in Mathematics*, Vol. 220. Springer-Verlag, Berlin-New York (1971)
- [41] Corduneanu, C.: *Integral equations and stability of feedback systems*. Academic Press, New York (1973)
- [42] Csizmadia, L., Hatvani, L.: *An extension of the Levi-Weckesser method to the stabilization of the inverted pendulum under gravity*. *Meccanica. International Journal of Theoretical and Applied Mechanics* **49**(5), 1091–1100 (2014)
- [43] Davis, J.M., Henderson, J., Wong, P.J.Y.: *General Lidstone problems: multiplicity and symmetry of solutions*. *Journal of Mathematical Analysis and Applications* **251**(2), 527–548 (2000)
- [44] De Coster, C., Habets, P.: *An overview of the method of lower and upper solutions for ODEs*. In: *Nonlinear Analysis and Its Applications to Differential Equations*, vol. 43, pp. 3–22. Birkhauser, Boston (2001)

-
- [45] De Coster, C., Habets, P.: *The lower and upper solutions method for boundary value problems*. In: Handbook of Differential Equations, pp. 69–160. Elsevier/North-Holland, Amsterdam (2004)
- [46] De Coster, C., Habets, P.: Two-point boundary value problems: lower and upper solutions. Elsevier (2006)
- [47] Derrick, G.H.: *Comments on nonlinear wave equations as models for elementary particles*. Journal of Mathematical Physics **5**(9), 1252–1254 (1964)
- [48] Djebali S.; Guedda, L.: *A third order boundary value problem with nonlinear growth at resonance on the half-axis*. Mathematical Methods in the Applied Sciences (2016)
- [49] Došlá, Z., Marini, M., Matucci, S.: *A boundary value problem on a half-line for differential equations with indefinite weight*. Communications in Applied Analysis **15**(2-4), 341–352 (2011)
- [50] Drábek, P., Holubová, G., Matas, A., Nečesal, P.: *Nonlinear models of suspension bridges: Discussion of results*. Applied Mathematics **48**, 497–514 (2003)
- [51] Drábek, P., Manásevich, R.: *On the closed solution to some nonhomogeneous eigenvalue problems with p -Laplacian*. Differential and Integral Equations. An International Journal for Theory & Applications **12**(6), 773–788 (1999)
- [52] Du, Y.: Order structure and topological methods in nonlinear partial differential equations: Maximum principles and applications. World Scientific (2006)
- [53] Feng, W., Webb, J.: *Solvability of three point boundary value problems at resonance*. Nonlinear Analysis: Theory, Methods & Applications **30**(6), 3227–3238 (1997)
- [54] Fialho, J., Minhós, F., Carrasco, H.: *Singular and classical second order ϕ -Laplacian equations on the half-line with functional boundary conditions*. Electronic Journal of Qualitative Theory of Differential Equations **2017**(10), 1–15 (2017)
- [55] Fife, P.C.: Mathematical aspects of reacting and diffusing systems, *Lecture Notes in Biomathematics*, vol. 28. Springer-Verlag, Berlin-New York (1979)
- [56] Figueroa, R., Tojo, F.A.F.: *Fixed points of Hammerstein-type equations on general cones*. Fixed Point Theory **19**(2), 571–586 (2016)
-

- [57] Franco, D., Infante, G., O'Regan, D.: *Nontrivial solutions in abstract cones for Hammerstein integral systems*. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **14**(6), 837–852 (2007)
- [58] Franco, D., Infante, G., Zima, M.: *Second order nonlocal boundary value problems at resonance*. Mathematische Nachrichten **284**(7), 875–884 (2011)
- [59] Frigon, M., Pouso, R.L.: *Theory and applications of first-order systems of Stieltjes differential equations*. Advances in Nonlinear Analysis (2016)
- [60] Gouin, H., Rotoli, G.: *An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids*. Mechanics Research Communications **24**(3), 255–260 (1997)
- [61] Graef, J.R., Kong, L., Minhós, F.: *Generalized Hammerstein equations and applications*. Results in Mathematics **72**, 369–383 (2017)
- [62] Graef, J.R., Kong, L., Wang, H.: *Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem*. Journal of Differential Equations **245**(5), 1185–1197 (2008)
- [63] Graef, J.R., Kong, L., Wang, H.: *A periodic boundary value problem with vanishing Green's function*. Applied Mathematics Letters **21**(2), 176–180 (2008)
- [64] Granas, A., Dugundji, J.: Fixed point theory. Springer (2003)
- [65] Greguš, M.: Third order linear differential equations, *Mathematics and its Applications (East European Series)*, vol. 22. D. Reidel Publishing Co., Dordrecht (1987)
- [66] Grossinho, M.d.R., Minhós, F., Tersian, S.: *Positive homoclinic solutions for a class of second order differential equations*. Journal in Mathematical Analysis and Applications **240**(1), 163–173 (1999)
- [67] Grossinho, M.R., Minhós, F., Tersian, S.: *Homoclinic and periodic solutions for some classes of second order differential equations*. In: Nonlinear analysis and its applications to differential equations, pp. 289–298. Birkhauser, Boston, MA (2001)
- [68] Guo, D., Lakshmikantham, V.: Nonlinear problems in abstract cones. Academic press (2014)
- [69] Guo, L., Sun, J., Zhao, Y.: *Existence of positive solutions for nonlinear third-order three-point boundary value problems*. Nonlinear Analysis: Theory, Methods & Applications **68**, 3151–3158 (2008)

-
- [70] Hakl, R., Torres, P.J.: *Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign*. Applied Mathematics and Computation **217**(19), 7599–7611 (2011)
- [71] Hammerstein, A.: *Nichtlineare integralgleichungen nebst anwendungen*. Acta Mathematica **54**, 117–176 (1930)
- [72] Han, X.: *Positive solutions for a three-point boundary value problem at resonance*. Journal of Mathematical Analysis and Applications **336**(1), 556–568 (2007)
- [73] Hardy, G.H.: *Orders of infinity, the ‘Infinitärrechnung’ of Paul Du Bois-Reymond*. Cambridge University Press (1910)
- [74] Heikkilä, S., Lakshmikantham, V.: *Monotone iterative techniques for discontinuous nonlinear differential equations, Monographs and Textbooks in Pure and Applied Mathematics*, vol. 181. Marcel Dekker, Inc., New York (1994)
- [75] Hewitt, E., Stromberg, K.: *Real and abstract analysis*. Springer-Verlag, New York-Heidelberg (1975)
- [76] Hill, G.W.: *On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon*. Acta Mathematica **8**(1), 1–36 (1886)
- [77] Holmes, M.H.: *Introduction to perturbation methods*, vol. 20. Springer Science & Business Media (2012)
- [78] Infante, G., Pietramala, P.: *A third order boundary value problem subject to nonlinear boundary conditions*. Academy of Sciences of the Czech Republic. Mathematical Institute. Mathematica Bohemica **135**(2), 113–121 (2010)
- [79] Infante, G., Pietramala, P., Tojo, F.A.F.: *Nontrivial solutions of local and non-local Neumann boundary value problems*. Proceedings of the Edinburgh Mathematical Society: Section A **2**(146), 337–369 (2016)
- [80] Infante, G., Webb, J.: *Nonzero solutions of Hammerstein integral equations with discontinuous kernels*. Journal of Mathematical Analysis and Applications **272**(1), 30–42 (2002)
- [81] Infante, G., Webb, J.: *Three point boundary value problems with solutions that change sign*. Journal of Integral Equations and Applications **15**, 37–57 (2003)
-

- [82] Jaroš, J., Takaši, K., Manojlović, J.: *Asymptotic analysis of positive solutions of generalized Emden-Fowler differential equations in the framework of regular variation*. Central European Journal of Mathematics **11**(12), 2215–2233 (2013)
- [83] Jiang, C., Zhai, C.: *Existence of nontrivial solutions for a nonlinear fourth-order boundary value problem via iterative method*. Journal of Nonlinear Sciences and Applications **9**, 4295–4304 (2016)
- [84] Jiang, W.: *Solvability for p -Laplacian boundary value problem at resonance on the half-line*. Boundary Value Problems **2013**(1), 207 (2013)
- [85] Jiang, W., Yang, C.: *The existence of positive solutions for multi-point boundary value problem at resonance on the half-line*. Boundary Value Problems **2016**:13, 1–12 (2016)
- [86] Jurkiewicz, M.: *Existence result for the Lidstone boundary value problem at resonance*. Journal of Mathematical Analysis and Applications **394**, 248–259 (2012)
- [87] Karlin, S.: *The existence of eigenvalues for integral operators*. Transactions of the American Mathematical Society **113**, 1–17 (1964)
- [88] Kaufmann, E.R.: *Existence of solutions for a nonlinear second-order equation with periodic boundary conditions at resonance*. Communications in Applied Analysis **18**, 163–174 (2014)
- [89] Kelley, J.L.: General topology, vol. 27. Springer Science & Business Media (1975)
- [90] Kiguradze, I.T.: Some singular boundary value problems for ordinary differential equations. Izdat. Tbilis. Univ., Tbilisi (1975)
- [91] Kiguradze, I.T., Chanturia, T.A.: Asymptotic properties of solutions of non-autonomous ordinary differential equations, *Mathematics and its Applications (Soviet Series)*, vol. 89. Kluwer Academic Publishers Group, Dordrecht (1993). Translated from the 1985 Russian original
- [92] Kiguradze, I.T., Shekhter, B.L.: *Singular boundary value problems for second-order ordinary differential equations*. In: Current problems in mathematics. Newest results, Vol. 30 (Russian), Itogi Nauki i Tekhniki, pp. 105–201, 204. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1987). Translated in J. Soviet Math. **43** (1988), no. 2, 2340–2417

-
- [93] Kim, C.G.: *Existence and iteration of positive solutions for multi-point boundary value problems on a half-line*. Computers & Mathematics with Applications **61**(7), 1898–1905 (2011)
- [94] Kosmatov, N.: *Multi-point boundary value problems on an unbounded domain at resonance*. Nonlinear Analysis: Theory, Methods & Applications **68**(8), 2158–2171 (2008)
- [95] Kuang, Y.: *Delay differential equations: with applications in population dynamics*, vol. 191. Academic Press (1993)
- [96] Ladde, G.S., Lakshmikantham, V., Vatsala, A.S.: *Monotone iterative techniques for nonlinear differential equations, Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics*, vol. 27. Pitman (Advanced Publishing Program), Boston, MA; distributed by John Wiley & Sons, Inc., New York (1985)
- [97] Lazer, A.C., McKenna, P.J.: *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*. SIAM **32**, 537–578 (1990)
- [98] Levi, M., Weckesser, W.: *Stabilization of the inverted linearized pendulum by high frequency vibrations*. SIAM Review. A Publication of the Society for Industrial and Applied Mathematics **37**(2), 219–223 (1995)
- [99] Lian, H., Ge, W.: *Solvability for second-order three-point boundary value problems on a half-line*. Applied Mathematics Letters **19**(10), 1000–1006 (2006)
- [100] Linde, A.P.: *Particle physics and inflationary cosmology*. Hardwood Academic, Chur, Switzerland (1990)
- [101] Lloyd, N.G.: *Degree theory*. Cambridge University Press (1978)
- [102] López Somoza, L., Minhós, F.: *Existence and multiplicity results for some generalized Hammerstein equations with a parameter*. arXiv preprint arXiv:1811.06118
- [103] López-Somoza, L., Minhós, F.: *On multi-point resonant problems on the half-line*. arXiv preprint arXiv:1811.06117
- [104] Ma, R.: *Nonlinear periodic boundary value problems with sign-changing Green's function*. Nonlinear Analysis: Theory, Methods & Applications **74**(5), 1714–1720 (2011)
-

- [105] Ma, R., Yang, Y.: *Existence result for a singular nonlinear boundary value problem at resonance*. Nonlinear Analysis: Theory, Methods & Applications **68**(3), 671–680 (2008)
- [106] Magnus, W., Winkler, S.: *Hill's equation*. Dover Publications, Inc., New York (1979). Corrected reprint of the 1966 edition
- [107] Marini, M., Matucci, S.: *A boundary value problem on the half-line for superlinear differential equations with changing sign weight*. Rendiconti dell'Istituto di Matematica dell'Università di Trieste **44**, 117–132 (2012)
- [108] Matucci, S., Řehák, P.: *Asymptotics of decreasing solutions of coupled p -Laplacian systems in the framework of regular variation*. Annali di Matematica Pura ed Applicata **193**(3), 837–858 (2014)
- [109] Mawhin, J.: *Points fixes, points critiques et problèmes aux limites*. Séminaire de Mathématiques Supérieures **92**, 162 pp. (1985). (in French)
- [110] Mawhin, J.: *Twenty years of ordinary differential equations through twelve Oberwolfach meetings*. Results in Mathematics **21**(1-2), 165–189 (1992)
- [111] Mawhin, J.: *Boundary value problems for nonlinear ordinary differential equations: from successive approximations to topology*. In: Development of mathematics 1900–1950, basel edn., pp. 443–477. Birkhauser (1994)
- [112] Mawhin, J.: *Bounded solutions of nonlinear ordinary differential equations*. In: Non-linear analysis and boundary value problems for ordinary differential equations, pp. 121–147. Springer, Vienna (1996)
- [113] McShane, E.J.: *Integration*. Princeton University Press (1947)
- [114] Minhós, F., Carrasco, H.: *Existence of homoclinic solutions for nonlinear second-order problems*. Mediterranean Journal of Mathematics **13**: 3849 (2016)
- [115] Minhós, F., Gyulov, T., Santos, A.I.: *Lower and upper solutions for a fully nonlinear beam equations*. Nonlinear Analysis **71**, 281–292 (2009)
- [116] Minhós, F., Sousa, R.: *On the solvability of third-order three point systems of differential equations with dependence on the first derivative*. Bulletin of the Brazilian Mathematical Society, New Series **3**, 485–503 (2016)
- [117] Minhós F.; Carrasco, H.: *Unbounded solutions for functional problems on the half-line*. Abstract and Applied Analysis (2016)

-
- [118] Minhós F.; Carrasco, H.: *Homoclinic solutions for nonlinear general fourth-order differential equations*. Mathematical Methods in the Applied Sciences (2017)
- [119] O'Regan, D., Zima, M.: *Leggett-Williams norm-type fixed point theorems for multivalued mappings*. Applied Mathematics and Computation **187**(2), 1238–1249 (2007)
- [120] Pouso, R.L., Rodríguez, A.: *A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives*. Real Analysis Exchange **40**(2), 1–35 (2015)
- [121] Przeradzki, B.: *The existence of bounded solutions for differential equations in Hilbert spaces*. Annales Polonici Mathematici **LVI.2**, 103–121 (1992)
- [122] Rachůnková, I., Rachůnek, L.: *Asymptotic formula for oscillatory solutions of some singular nonlinear differential equation*. Abstract and Applied Analysis **2011**(1), 1–9 (2011)
- [123] Rachůnková, I., Rachůnek, L., Tomeček, J.: *Existence of oscillatory solutions of singular nonlinear differential equations*. Abstract and Applied Analysis **2011**(1), 1–20 (2011)
- [124] Rachůnková, I., Staněk, S.: *Topological degree method in functional boundary value problems*. Nonlinear Analysis: Theory, Methods & Applications **27**(2), 153–166 (1996)
- [125] Rachůnková, I., Tomeček, J.: *Homoclinic solutions of singular nonautonomous second-order differential equations*. Boundary Value Problems **2009**(1), 959,636 (2009)
- [126] Rachůnková, I., Tomeček, J.: *Bubble-type solutions of nonlinear singular problems*. Mathematical and Computer Modelling **51**(5-6), 658–669 (2010)
- [127] Rachůnková, I., Tomeček, J.: *Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics*. Nonlinear Analysis: Theory, Methods & Applications **72**(3-4), 2114–2118 (2010)
- [128] Rachůnková, I., Tomeček, J., Stryja, J.: *Oscillatory solutions of singular equations arising in hydrodynamics*. Advances in Difference Equations **2010**(1), 872,160 (2010)
-

- [129] Reed, M., Simon, B.: *Methods of modern mathematical physics. I*, second edn. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York (1980). Functional analysis
- [130] Rohleder, M.: *On the existence of oscillatory solutions of the second order nonlinear ODE*. Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **51**(2), 107–127 (2012)
- [131] Rohleder, M., Burkotová, J., López-Somoza, L., Stryja, J.: *On unbounded solutions of singular IVPs with ϕ -Laplacian*. Electronic Journal of Qualitative Theory of Differential Equations **2017**(80), 1–26 (2017)
- [132] Schauder, J.: *Der fixpunktsatz in funktionalriiumen*. Studia Mathematica **2**, 171–180 (1930)
- [133] Simmons, G.F.: *Differential equations with applications and historical notes*. International Series in Pure and Applied Mathematics. McGraw-Hill (1972)
- [134] Singh, R.K., Manhas, J.S.: *Composition operators on function spaces*, vol. 179. Elsevier (1993)
- [135] Sipser, M.: *Introduction to the theory of computation*, vol. 2. Thomson Course Technology Boston (2006)
- [136] Stromberg, K.R.: *An introduction to classical real analysis*. Chapman & Hall (1996)
- [137] Takaši, K., Manojlović, J.V., Milošević, J.: *Intermediate solutions of second order quasilinear ordinary differential equations in the framework of regular variation*. Applied Mathematics and Computation **219**(15), 8178–8191 (2013)
- [138] Talenti, G.: *Best constant in Sobolev inequality*. Ann. Mat. Pura Appl. **4**, 353–372 (1976)
- [139] Talvila, E.: *Necessary and sufficient conditions for differentiating under the integral sign*. American Mathematical Monthly **108**(5), 432–436 (2001)
- [140] Talvila, E.: *Some divergent trigonometric integrals*. American Mathematical Monthly **108**(6), 544–548 (2001)
- [141] Timoshenko, S.P.: *Theory of elastic stability*. McGraw-Hill, New York (1961)
- [142] Torres, P.J.: *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnosel'skiĭ fixed point theorem*. J. Differential Equations **190**(2), 643–662 (2003)

-
- [143] Torres, P.J., Zhang, M.: *A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle*. *Mathematische Nachrichten* **251**, 101–107 (2003)
- [144] Vampolová, J.: *On existence and asymptotic properties of Kneser solutions to singular second order ODE*. *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica* **52**(1), 135–152 (2013)
- [145] Wang, M.X., Cabada, A., Nieto, J.: *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions*. *Annales Polonici Mathematici* **58**, 221–235 (1993)
- [146] Wang, Y.M.: *On 2nth-order Lidstone boundary value problems*. *Journal of Mathematical Analysis and Applications* **312**(2), 383–400 (2005)
- [147] Webb, J.: *Boundary value problems with vanishing Green's function*. *Communications in Applied Analysis* **13**(4), 587–595 (2009)
- [148] Webb, J.: *A class of positive linear operators and applications to nonlinear boundary value problems*. *Topological Methods in Nonlinear Analysis* **39**(2), 221–242 (2012)
- [149] Webb, J., Infante, G.: *Positive solutions of nonlocal boundary value problems: a unified approach*. *Journal of the London Mathematical Society* **74**(3), 673–693 (2006)
- [150] Webb, J., Infante, G., Franco, D.: *Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions*. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **138**(02), 427–446 (2008)
- [151] Webb, J., Lan, K.: *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type*. *Topological Methods in Nonlinear Analysis* **27**, 91–115 (2006)
- [152] Webb, J., Zima, M.: *Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems*. *Nonlinear Analysis* **71**(3), 1369–1378 (2009)
- [153] Whitney, H.: *Analytic extensions of differentiable functions defined in closed sets*. *Transactions of the American Mathematical Society* **36**, 63–89 (1934)
-

- [154] Xu, J., Yang, Z.: *Positive solutions for singular Sturm-Liouville boundary value problems on the half line*. Electronic Journal of Differential Equations **2010**(171), 1–8 (2010)
- [155] Yang, A., Henderson, J., Wang, D.: *The study of higher-order resonant and non-resonant boundary value problems*. Electronic Journal of Qualitative Theory of Differential Equations **2016**(3), 1–10 (2016)
- [156] Yang, L., Zhang, W., Liu, X., Shen, C., Chen, H.: *Positive solutions for a nonlinear third order multipoint boundary value problem*. Pacific Journal of Mathematics **249**(1), 177–188 (2011)
- [157] Zabreiko, P.P., Krasnoselskii, M.A., Stetsenko, V.Y.: *Bounds for the spectral radius of positive operators*. Matematicheskie Zametki **1**, 461–468 (1967)
- [158] Zettl, A.: *Sturm-Liouville theory*, *Mathematical Surveys and Monographs*, vol. 121. American Mathematical Society, Providence, RI (2005)
- [159] Zhang, B., Liu, X.: *Existence of multiple symmetric positive solutions of higher order Lidstone problems*. Journal of Mathematical Analysis and Applications **284**(2), 672–689 (2003)
- [160] Zhang, M.: *Certain classes of potentials for p -Laplacian to be non-degenerate*. Mathematische Nachrichten **278**(15), 1823–1836 (2005)
- [161] Zhang, M.: *Optimal conditions for maximum and antimaximum principles of the periodic solution problem*. Boundary Value Problems **2010**, 1–26 (2010)
- [162] Zhang, M., Li, W.: *A Lyapunov-type stability criterion using L^α norms*. Proceedings of the American Mathematical Society **130**(11), 3325–3333 (2002)
- [163] Zhong, S., An, Y.: *Existence of positive solutions to periodic boundary value problems with sign-changing Green's function*. Boundary Value Problems **2011**:8, 1–6 (2011)
- [164] Zima, M.: *Applications of the spectral radius to some integral equations*. Commentationes Mathematicae Universitatis Carolinae **36**(4), 695–703 (1995)